DEBT SERVICE AND FOREIGN ASSISTANCE:
AN ANALYSIS OF PROBLEMS AND PROSPECTS IN
LESS DEVELOPED COUNTRIES*

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Debt Service and Foreign Assistance: An Analysis of Problems and Prospects in Less Developed Countries

I. Introduction

The large increase in foreign assistance to the less developed world in the late 1950's and early 1960's has created a set of problems concerning the repayment of foreign debts which promise to become increasingly severe throughout the next decade. The purpose of this paper is to draw on recent experience in handling problems of debt service in order to make some projections and assess future prospects.

The first part of this paper points out some trends in debt rescheduling exercises which have important implications for future policy. Section II discusses the causes of debt servicing difficulties. Data on debt service and other related economic variables are used to perform a statistical discriminant analysis to distinguish the characteristics of those countries which experience severe problems and those which do not. Traditional discriminant analysis techniques are not quite adequate to this task. A modified approach based on the assumption of unequal variances of variables among different categories of countries is applied to shed further light on the distinguishing characteristics of problem countries. The modified discriminant analysis techniques are elaborated in a statistical appendix by the senior author. The next section uses past data on the level and structure of foreign debt for 17 less developed countries to project debt service payments over the next decade. These projections are then tied to a projection of the discriminant function to assess the likely frequency of future debt servicing problems. Finally, a concluding section assesses
the policy implications of the preceding analysis, especially with regard to the terms of foreign assistance.

II. Trends in Debt Rescheduling

2.1 Institutional Arrangements

In the last ten years, a number of countries experienced such severe difficulties in servicing their debt that they negotiated with creditors to postpone payments of interest or principal. In some cases these negotiations were preceded by a period in which arrears of payments occurred. There have been at least 20 cases of debt reschedulings over the last ten years for ten different countries. These are listed in Table I.

The great majority of reschedulings have occurred in multilateral settings. Only Yugoslavia, Liberia and the U.A.R. negotiated bilaterally with each major creditor. The Latin American reschedulings took place under the auspices of the Hague and Paris Clubs, groups of creditors formed originally for the purpose of pooling non-convertible currencies. France was most often the major creditor. The Ghanaian negotiations were held in London with IMF sponsorship. The United Kingdom and the Federal Republic of Germany were the major Ghanaian creditors and virtually dictated the terms of debt rollover. The Turkey, Indonesia, and India negotiations were conducted within the framework of consortia of the major aid-giving countries. In the India case, the leadership in the negotiations stemmed from the IBRD.

The IMF played a significant role in nearly all of the reschedulings listed in Table I. Typically, the debtor received some IMF standby credits in conjunction with debt rollover. The acceptance of standby credits implied
an obligation on the part of the debtor to fulfill certain pledges with regard to monetary and fiscal policy. Frequently, the creditors have also required the debtor to limit its future borrowing of short-term commercial credit.

The Latin American and Ghana negotiations are most often described as "ad hoc informal meetings of the major creditors." The creditor countries in these negotiations have done their best to maintain the idea that debt relief is not an institution but a very serious and unique event whenever it occurs. The recent trend toward the use of consortia, which are also responsible for the pledging and coordination of the regular flows of financial aid, has resulted in the erosion of the ad hoc concept. India, Indonesia, and Turkey are now in a situation which promises periodic rescheduling meetings.
<table>
<thead>
<tr>
<th>Country</th>
<th>Years in which Payments Deferred</th>
<th>Institutional Arrangement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>1957(^1)</td>
<td>Paris Club</td>
</tr>
<tr>
<td></td>
<td>1961-62</td>
<td>Paris Club</td>
</tr>
<tr>
<td></td>
<td>1963-64</td>
<td>Paris Club</td>
</tr>
<tr>
<td></td>
<td>1965</td>
<td>Paris Club</td>
</tr>
<tr>
<td>Brazil</td>
<td>1961-65</td>
<td>The Hague Club</td>
</tr>
<tr>
<td></td>
<td>1964-65</td>
<td>The Hague Club</td>
</tr>
<tr>
<td>Chile</td>
<td>1965-66</td>
<td>Paris Club</td>
</tr>
<tr>
<td>Turkey</td>
<td>1953-1963</td>
<td>OEEC Auspices</td>
</tr>
<tr>
<td></td>
<td></td>
<td>OECD Donor Consortium</td>
</tr>
<tr>
<td>Indonesia</td>
<td>1966-67</td>
<td>Donor Consortium</td>
</tr>
<tr>
<td></td>
<td>1968</td>
<td>Donor Consortium</td>
</tr>
<tr>
<td></td>
<td>1969</td>
<td>Donor Consortium</td>
</tr>
<tr>
<td>India</td>
<td>1968</td>
<td>Donor Consortium</td>
</tr>
<tr>
<td></td>
<td></td>
<td>under IBRD leadership</td>
</tr>
<tr>
<td>Ghana</td>
<td>1966-68</td>
<td>IMF Auspices</td>
</tr>
<tr>
<td></td>
<td>1969-70</td>
<td>IMF Auspices</td>
</tr>
<tr>
<td></td>
<td>1960-72</td>
<td>IMF Auspices</td>
</tr>
<tr>
<td>Peru</td>
<td>1968-69</td>
<td>United Kingdom Sponsorship</td>
</tr>
<tr>
<td>Liberia</td>
<td>1963</td>
<td>Bilateral</td>
</tr>
<tr>
<td>U.A.R.</td>
<td>1967-68(^2)</td>
<td>Bilateral*</td>
</tr>
<tr>
<td>Yugoslavia</td>
<td>1965-66</td>
<td>Bilateral</td>
</tr>
</tbody>
</table>

\*Rescheduled arrears with major creditors except the United States.

\(^1\)/ A number of countries not included in this table rescheduled some very short-term debt (less than one year maturity).

\(^2\)/ Dates of agreements to reschedule arrears.
The difference in conception stems in part from differences in the structure of the debt. Many of the Latin American negotiations arose from difficulties encountered in servicing short- and medium-term commercial debt. The more recent phenomenon faced by consortia creditors is one in which long-term official lending forms a much more significant role in the debt-service burden. A rescheduling of payments over a one to five year period combined with some restrictions on the volume of commercial borrowing can significantly reduce the amount of debt service in the former case. In the long-term lending case, the debt must be rescheduled over a considerable period of time to have any significant impact on the debt-service burden. In the Indian case there was no balance of payments crisis but it was seen easily that the level of debt service was likely to cause problems in the future. The goal of the consortia-approved development program was the reduction of debt service to about 20 per cent of exports. This necessitated a rescheduling of about 25 per cent of the debt due in 1969 but will require regular and increasing amounts of rescheduling for quite some time to come.

2.2 Amounts and Terms of Rescheduling

The amount of debt service rescheduled in the last decade has been considerable, probably in the order of $2 billion. A partial list is contained in Table II. The moratoriums on payment of debt service are typically very short. In the earlier rescheduling, this reflected the ad hoc nature of the negotiations and the hope that the difficulties would be of short duration.1/ In the cases of India,

1/On the other hand, the list of debt reschedulings in Table I indicate that recurrence of debt problems after a debt rollover are quite frequent. The short moratoriums frequently do nothing more than shift the burden of debt service one or two years into the future.
Indonesia, and Turkey, the shortness of the moratoriums may be a simple case of the reluctance to change traditional formulae but also may indicate a preference for repeated reschedulings to maintain control over the debtor countries’ economic policies.

Table II

AMOUNTS OF DEBT RESCHEDULINGS

<table>
<thead>
<tr>
<th>Country</th>
<th>Year</th>
<th>Moratorium Length (years)</th>
<th>Amount ($ million)</th>
<th>Portion of Service Due (per cent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>India</td>
<td>1968</td>
<td>1</td>
<td>100</td>
<td>25(^{a})</td>
</tr>
<tr>
<td>Ghana</td>
<td>1966-1968</td>
<td>2(\frac{1}{4})</td>
<td>170</td>
<td>80(^{a}), (^{e})</td>
</tr>
<tr>
<td>Turkey</td>
<td>1965-1967</td>
<td>3</td>
<td>217</td>
<td>80/60(^{b}), (^{c}), (^{e})</td>
</tr>
<tr>
<td>Chile</td>
<td>1965-1966</td>
<td>2</td>
<td>90</td>
<td>70(^{b}), (^{e})</td>
</tr>
<tr>
<td>Brazil</td>
<td>1964-1965</td>
<td>2</td>
<td>190</td>
<td>70(^{a}), (^{e})</td>
</tr>
<tr>
<td>Argentina</td>
<td>1965</td>
<td>1</td>
<td>90</td>
<td>60(^{b}), (^{e})</td>
</tr>
<tr>
<td>Indonesia</td>
<td>1966-1967</td>
<td>1(\frac{1}{4})</td>
<td>350</td>
<td>100(^{a}), (^{d})</td>
</tr>
</tbody>
</table>

a. Interest plus principal.
b. Principal only.
c. 60 per cent commercial and 30\% per cent official debt.
d. Excluding Eastern Bloc debt.
e. Excluding certain official credits.

The terms have varied considerably. Table III summarizes some recent experience. It is hard to find meaningful invocation of general principles although the relatively soft terms for India and Turkey presumably reflect the long-run nature of their debt servicing problems. The traditional formula for negotiations is to decide at meetings attended by all creditors on the amounts rescheduled, the length of the moratorium, and the period of repayment. The consolidation interest rates typically differ from creditor to creditor, being determined on the basis of bilateral negotiations. Frequently, the result of these negotiations has been the use of a so-called commercial credit rate, often 5 or 6 per cent and at times higher than 8 per cent.
2.3 Burden Sharing

The Anglo-Saxon legal tradition of bankruptcy and debt settlement does not generally pretend to find, in different types of borrowing, the "cause" of the whole problem. Each creditor is expected to absorb an equal burden in terms of the portions of total debt owed him which is not paid. On the international plane, there is often considerable disparity between the terms of different donors' lending, with the result that there has been a fair amount of concern over the appropriate shares of creditors when an LDC needs help. The U.S. in particular, as a soft lender, has been in conflict with the hard-lending Europeans. In spite of the fact that its share of the total rescheduling is usually small by any measure, the U.S. has been moderately successful in having long-term debt and interest excluded from reschedulings. Outside of this, however, justice has most often been equated with relief by each creditor of an equal percentage of the total amount due it. The Indian case marks an important breakthrough. The IBRD devised a formula for determining the relief to be given by each creditor in such a way that the hard lenders would have to sacrifice more. It was decided how much India could afford to pay per year (around 20 per cent of exports), and this was divided by total debt outstanding. This came to 6 per cent. Every creditor was then to reschedule whatever service payments were over this limit. Countries under the limit (like the U.S.) got no rebate and, in fact, had to make a certain minimum contribution to the rescheduling. It seems likely that this idea will be used more often in the future, although as rescheduling comes to be considered more equivalent to aid, the criteria now being considered for distributing the aid burden, such as per capita income, may enter the picture.
Table III

TERMS OF RESCHEDULINGS

<table>
<thead>
<tr>
<th>Country</th>
<th>Year</th>
<th>Grace Period (years)</th>
<th>Repayment Period (years)</th>
<th>Interest Rate (per cent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>India</td>
<td>1968</td>
<td>10</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Ghana</td>
<td>1966-1968</td>
<td>2½</td>
<td>8</td>
<td>Bilateral determination</td>
</tr>
<tr>
<td>Turkey</td>
<td>1965-1967</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Chile</td>
<td>1965</td>
<td>3</td>
<td>5</td>
<td>Bilateral determination</td>
</tr>
<tr>
<td>Brazil</td>
<td>1964-1965</td>
<td>3</td>
<td>5</td>
<td>Bilateral determination</td>
</tr>
<tr>
<td>Argentina</td>
<td>1965</td>
<td>3</td>
<td>5</td>
<td>Bilateral determination</td>
</tr>
<tr>
<td>Indonesia</td>
<td>1968</td>
<td>3</td>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>

- Most common terms

III. Causes of Debt-Servicing Problems

3.1 Debt Service and Balance of Payments

It is difficult to speak of the causes of reschedulings without reference to the level of generality involved. The cause at one level is usually a balance of payments crisis. Aside from cases in which individual private debtors default on their obligations, the inability to pay interest and principal on debt outstanding is just one among a number of other indications of a lack of foreign exchange. In one sense, the ability to pay debt is limited only by the extent to which foreign exchange can be saved from a very strict curtailment of imports, and foreign exchange can be earned by exporting as much of the domestic product as can be sold abroad. When debt service is very large, however, the monetary, fiscal, tariff, and exchange rate policies required to restore international balance might result in extreme sacrifices and political difficulties which both the debtor and creditor countries might wish to avoid. The rescheduling of debt service is alternative means of helping to alleviate a country's foreign exchange difficulties which avoids some of the difficulties associated with
more stringent economic policies in the debtor country. This is, of course, not the only means available, and debt rescheduling ought to be viewed in the context of other policy alternatives designed to meet foreign exchange shortages.

3.2 Special Difficulties with Debt Service Obligations

In one sense, at least, a high level of debt service presents its own special problems. Unlike payments for imports of goods and services, debt service payments are fixed obligations which cannot be avoided without severe repercussions on the foreign balance for years to come. Imports can be reduced temporarily merely by applying appropriate restrictions on demand. When foreign exchange is more freely available, restrictions on import demand can be lifted. The supply of goods and services for import is likely to be affected very little. When a country defaults on its debt, however, the supply of capital in the future will be severely reduced. Without any assurances of repayment, both official and private creditors will not be very willing to lend. Only in the most extreme circumstances will a country be willing to default. The unattractiveness of default is considerably strengthened by the existing web of international political relationships. A country with close economic and political ties to the United States or Britain might find default accompanied by a series of diplomatic and economic reprisals; e.g., cutting the sugar quota, limiting oil imports, elimination of Commonwealth preferences, etc. Thus, the U.A.R. and Indonesia under Sukarno found default with respect to the Western powers attractive in view of their attenuated relationships with these countries.
Leaving aside the difficult question of when it pays a country to default,\(^1\) the fixed nature of debt service obligations has focused the attention of many analysts and experts on the debt service ratio. This ratio is defined as the ratio of service on debt to export earnings.\(^2\)

The rationale for the use of the debt service ratio as an indicator of a country's debt-serving capacity is that an increase in the debt service ratio indicates increased vulnerability to foreign exchange crises. Any shortfall in foreign exchange earnings or capital imports which is not covered by exchange reserves must be met by reducing imports; since debt service is a fixed obligation, the higher the debt service ratio, the greater is the relative burden on import reduction for a given shortfall in foreign exchange.

Unfortunately, the debt service ratio in and of itself is not a very good indicator of a country's ability or lack of ability to pay its debts. The debt service ratio is merely an indicator of the proportion of foreign exchange earnings which are free to purchase imports. If exchange earnings are high relative to import demand, a high debt service ratio can be maintained. Furthermore, a country with good credit standing in international money markets may be able to finance a high debt service ratio, for a time at least, through a high level of borrowing.

The historical behavior of debt service ratios and instances of default also indicate an ability of some countries to tolerate high debt service ratios. Mexico and Israel have not defaulted nor requested

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\(^2\) Modified versions of the debt service ratio include the ratio of debt service to earnings from exports of goods and services and the ratio of debt service to current account foreign exchange receipts. The ratio of debt service plus payments of income from equity investments to various definitions of exchange earnings is also used.
debt rescheduling despite debt service ratios of 39 and 26 per cent respectively in recent years. 1/

Australia managed to avoid defaults on public and private debts with an investment service-exchange earnings ratio ranging from 43 to 44 per cent during the period 1930-1934. Canada avoided defaults and the imposition of exchange restrictions on current transactions with an investment service-exchange earnings ratio of 32 to 37 per cent over the 1931-1933 period. 2/

On the other hand, Bolivia, Brazil, Colombia, Cuba, Peru, and Uruguay defaulted in the period 1931-1933 with debt service ratios that were generally lower, in the order of 16 to 28 per cent. 3/

It is clear that there are a host of other factors which influence a country's ability to service debt. In previous studies a number of indicators other than the debt service ratio (denote by $X_1$) have been used as warning signals for debt servicing difficulties. Among these are:

(i) the rate of growth of exports ($X_2$),
(ii) the variability of export earnings ($X_3$),
(iii) "compressible" imports relative to non-compressible imports ($X_4$),
(iv) per capita income ($X_5$),
(v) the rate of amortization of outstanding debt ($X_6$),


2/ Raymond F. Mikesell, The Economics of Foreign Aid, Chicago, Aldine, 1968, p. 118. Investment service includes dividends on equity investments as well as debt service but the former typically very small relative to debt service for most less developed countries today.

(vi) the ratio of imports to GNP \((X_7)\)

(vii) the level of exchange reserves relative to imports \((X_8)\). 1

One can make a number of heuristic and theoretical arguments for the use of each and every one of these indicators, some convincing and others not so convincing. In order, however, to use these indicators for predictive purposes or for discriminating between countries which are likely to have debt-servicing problems and those which are not, one must specify some function of the various indicators which provides a composite index of debt servicing capacity.

3.3 Linear Discriminant Analysis

The composite index which we wish to estimate may be expressed as a linear function of the eight indicators above:

\[ Z = \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_8 X_8. \]

The methods for estimating the \(\beta\) weights are discussed in the appendix to this paper.

3.3a The Sample

The sample used to estimate the \(\beta\) 's is combined cross-country time series data. We chose the nine-year period 1960 to 1968. In order to reduce the number of observations, we eliminated from consideration all

1/See, for example, D. Avramovic, et al., Economic Growth and External Debt, Baltimore, Johns Hopkins Press, 1964, and OECD, D.A.C., Survey of Debt Situations, mimeo., 1966. In the latter document the Development Assistance Committee has suggested an "organic indicator of vulnerability" equal to the possible imports during a crisis, divided by current imports. Possible imports is defined as minimum expected exports, plus foreign exchange reserves, plus capital inflows, less debt service payments.

2/ There exists at least one unpublished attempt to compile a composite index of debt servicing capacity. See S.P. Magee, "An Empirical Study of Debt Servicing Capacity," mimeo. Magee estimates by regression analysis a composite index which combines the debt service ratio and the export growth rate.
countries smaller than Ghana, the smallest rescheduling country, in terms of population and gross national product. Other countries were eliminated because of lack of data. The result was to include 26 countries over 9 years or theoretically 234 observations. Absence of data for specific years reduced the number of observations to 145. In the period 1960/1968 we included 13 reschedulings in eight countries, Argentina, Brazil, Chile, Ghana, India, Indonesia, Turkey, and the U.A.R.

3.3 b The Indicators

The first indicator of debt servicing difficulty is the ratio of debt service to exports, $X_{1t}$. A one-year lag is specified in this and all subsequent $X$-variables, on the assumption that the debt service payment interruption in year $t$ occurs after decisions made near the end of year $t-1$ and that these decisions are based on the appearance of indicators during year $t-1$. Thus,

$$X_{1t} = \frac{S_{t-1}}{E_{t-1}}$$

(2)

where $S_{t-1}$ is debt service payments and $E_{t-1}$ is "normal" exports in year $t-1$. "Normal" exports are used rather than actual exports in year $t-1$, under the assumption that authorities pay little attention to temporary highs or lows in exports but base decisions on what normal exports can be expected to be. $E_{t-1}$ was calculated as the "predicted" exports in year $t-1$, based on a regression of the logarithm of exports on time for the five-year period ending in year $t-1$. One should note that the debt service ratio was calculated on the basis of data on public debt and publicly guaranteed private debt. Good data on private debt are not available.
The second indicator is \( X_2 \), the growth rate of exports. We assume that a country with a high export growth rate is less likely, \textit{ceteris paribus}, to reschedule since the prospects are brighter for increasing foreign exchange earnings in the near future. The growth rate of exports is calculated on the basis of four year averages, over an eight year period preceding the year of observation.

The variable \( X_3 \) is an export fluctuation index measured as the average absolute percentage deviation from an eight year trend preceding the year of observation. We reasoned that a country with stable export earnings was less vulnerable to foreign exchange crises and could tolerate a higher debt service ratio.

The fourth variable \( X_4 \) is "non-compressible imports" as a fraction of total imports. It represents the degree to which imports may be reduced in time of balance-of-payments crisis. The higher this value, the more difficult it will be for a country to meet a debt servicing burden, and, therefore, the more likely debt rescheduling. Non-compressible imports were essentially intermediate goods, capital goods, and basic foodstuffs. They were found by subtracting from total imports the values of the following "compressible" items: finished manufactured goods, meats, poultry, fruits, and nuts.

The fifth indicator \( X_5 \) is per capita income. It would seem likely that the lower per capita income, the less flexibility there would be for reducing consumption and thus, the more likely debt rescheduling.

The sixth indicator \( X_6 \) is the ratio of debt amortization to total outstanding debt (the inverse of the "average" maturity of loans). A low value for this indicator should be associated with a tendency to reschedule debt given the value of the debt service ratio and the other indicators.
First, a low amortization rate indicates that the current debt service ratio is unlikely to be reduced much in the near future even if no new borrowing takes place. A country in such a situation (e.g., India or Pakistan) may be tempted to try to reschedule in anticipation of future difficulties and in an attempt to increase the near term ability to incur additional debt. Secondly, a high amortization rate usually indicates a heavy reliance on commercial credit facilities, access to which is usually granted to the more 'credit-worthy' countries in the eyes of commercial lenders. A good credit reputation enhances a country's ability to obtain additional credits when shortfalls in exchange earnings occur. The ability to gain additional credits helps to avert foreign exchange crises and the necessity for debt rescheduling.

The seventh variable \( x_7 \) is the ratio of imports to Gross National Product. A country with low imports relative to GNP is more likely to be able to withstand temporary import cuts than a country with high imports relative to GNP. Since \( x_4 \) already accounts for consumption 'import compressibility,' \( x_7 \) may be thought of in the following terms. A country with high imports of intermediate inputs relative to GNP will find its production much more seriously threatened by inability to import than will a country which draws little of its intermediate inputs from imports. In sum, the higher \( x_3 \), the more likely a country is to require debt rescheduling.

Finally, the country's reserves must be considered. Variable \( x_8 \) is the ratio of imports to reserves, where reserves include gold reserves, holdings of dollars on sterling, and net position at the IMF.\(^1\) Other influence equal, the country with high reserves relative to imports is in less need of debt rescheduling.

\(^1\) That is, the ceiling of permissible borrowings less the amount of borrowings already incurred.
3.3c Data Sources

The data sources were as follows. Debt service, amortization, and outstanding debt were taken from studies by the IBRD\(^1\), Avramovic\(^2\), the Development Assistance Committee of the OECD\(^3\), and AID country files. An attempt was made to find "ex ante" debt service for the year in which rescheduling occurred. In cases where these data were available, they were used in place of debt service in the prior year: in variable \(x_{\cdot1}\), "expected" service in year \(t\) replaced observed service in year \(t-1\). \(^4\)

Population, dollar values of exports, imports, and reserves (gold and foreign exchange plus net IMF position) were taken from standard international statistical sources. Gross national product in current U.S. dollars was found from AID country files. The "non-compressible" imports were calculated from the SITC breakdowns of imports.\(^5\) (SITC 04) and raw materials and manufactured goods in SITC groups 2 through 7, excluding passenger cars, textile yarn, and finished paper goods. Thus, the consumer


\(^4\) That is, the use of debt service in year \(t-1\) to calculate \(x_{1t}\) was on the assumption that this value approximated the expected level of debt service in year \(t\) which would occur in the absence of rescheduling. Thus, in rescheduling cases in which the actual anticipated debt service for year \(t\) (if there were to be no rescheduling) was known, it was used in the calculation \(x_{1t}\).

\(^5\) U.N., Yearbook of International Trade Statistics, various years, For several countries, complete time series from 1957 to 1967 were not available. In these cases, data for the missing years were estimated by assuming that of total imports, the non-compressible fraction was identical to that for the closest three-year average for which data were available.
goods considered "compressible" were foods other than cereals, beverages and tobacco (SITC group 1), the exceptions to groups 2 through 7 mentioned above, and other manufactured goods (SITC group 3), such as clothing.

3.3d Empirical Results

The results of the discriminant analysis are shown in Table IV. Although the assumptions of regression analysis are not appropriate in discriminant analysis, we found it useful to apply the usual linear regression tests to obtain some notion of the relative importance of the various variables. The most striking result was the dominance of only three variables: the debt service ratio ($X_{d}$), the amortization/debt ratio ($X_{a}$) and the imports/reserves ratio ($X_{r}$). Only these three variables were statistically significant at the 5% level. Note that all three have coefficients with the "correct" sign. In model B, only these three variables were included, yet the multiple correlation coefficient declined very little.\footnote{Note that the $R^2$ cannot be interpreted as the "percent of variation explained," since even if the model perfectly predicted all observations as rescheduling or not rescheduling, the $R^2$ would not be unity because the estimated dependent variable is continuous but the observed "dependent" variable is zero or one.} Model C was estimated with only the debt service ratio and the amortization/debt ratio as explanatory indicators, on the grounds that a low reserve ratio and a rescheduling negotiation are symptoms of the same thing, a balance of payments problem. Furthermore, the debt service ratio and amortization/debt ratio are more susceptible to prediction than the imports/reserve ratio and thus model C lends itself more easily to a projection of debt servicing difficulties.

Table V shows the success of the linear discriminant functions in predicting the debt reschedulings in the data examined. The discriminant
Table IV

Linear Discriminant Functions

<table>
<thead>
<tr>
<th>Coefficient of:</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (constant)</td>
<td>-0.0891</td>
<td>-0.1068</td>
<td>-0.0045</td>
</tr>
<tr>
<td>$X_1$</td>
<td>1.4955</td>
<td>1.4204</td>
<td>1.6261</td>
</tr>
<tr>
<td></td>
<td>(5.16)</td>
<td>(6.46)</td>
<td>(7.07)</td>
</tr>
<tr>
<td>$X_2$</td>
<td>-0.0545</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-0.13)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_3$</td>
<td>-0.0724</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-0.11)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_4$</td>
<td>-0.1121</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-0.59)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_5$</td>
<td>0.0152</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(1.01)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_6$</td>
<td>-1.3739</td>
<td>-1.3168</td>
<td>-1.2147</td>
</tr>
<tr>
<td></td>
<td>(-3.27)</td>
<td>(-3.29)</td>
<td>(-5.23)</td>
</tr>
<tr>
<td>$X_7$</td>
<td>0.2655</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.70)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_8$</td>
<td>0.0360</td>
<td>0.0373</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(3.60)</td>
<td>(3.73)</td>
<td></td>
</tr>
<tr>
<td>$R_X$</td>
<td>0.3615</td>
<td>0.3501</td>
<td>0.2698</td>
</tr>
</tbody>
</table>

Note: Figures in parentheses are $t$ values.
functions in Table IV were applied to find the Z value of each observation, and the critical value Z* was determined by assuming that the probability of choosing from either the rescheduling or non-rescheduling population was equal and that the costs of misclassification were equal.

The linear discriminant functions correctly predicted rescheduling and non-rescheduling in all but 18 of the 145 country-years. The fact that models B and C perform as well as model A is consistent with the low coefficient significance for variables included in A but excluded in B and C. Concerning the prediction error, it is important to note that false predictions of rescheduling ("typeII" errors) were heavily concentrated in countries which had debt reschedulings in other years. Using model A, for example, 11 of the 15 type-II errors were for observations from countries which in other years did have reschedulings (Argentina, Brazil, Chile, India, Turkey). This pattern represents continuation of "bad" debt-service ratios and amortization rates in non-rescheduling years for countries that have recently rescheduled.

3.4 Discriminant Analysis with Unequal Variance

To improve the accuracy of classification of observations as "rescheduling" or "non-rescheduling," we may take into account differences in the degree of variability of the X-indicators between the two groups. The simple linear discriminant function above requires the assumption that the covariance matrix of the X-variables is identical for the rescheduling and non-rescheduling groups. In the Appendix, two methods are developed
Table V. Debt Rescheduling Predictions:
Linear Discriminant Functions

<table>
<thead>
<tr>
<th>Model</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I errors</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Type II errors</td>
<td>15</td>
<td>14</td>
<td>17</td>
</tr>
<tr>
<td>Total errors</td>
<td>18</td>
<td>17</td>
<td>10</td>
</tr>
<tr>
<td>Total observations</td>
<td>145</td>
<td>145</td>
<td>145</td>
</tr>
<tr>
<td>Critical value</td>
<td>.219</td>
<td>.239</td>
<td>.126</td>
</tr>
</tbody>
</table>

Note: Type I error = Predict non-rescheduling for rescheduling country-year
Type II error = Predict rescheduling for non-rescheduling country-year.

for estimation of discriminant functions when the covariance matrices differ for the two classes of observations. The first method estimates a quadratic function,

\( Z = X'AX + EX + c \)

where \( Z \) is the discriminant value for an observation, \( X \) is the observations vector of \( X \) variables, and \( A \) is an estimated matrix of coefficients, \( E \) an estimated vector of coefficients, and \( c \) an estimated constant.

The second method suggested in the Appendix is an iterative estimate of a linear discriminant function, in which the initial estimate assumes equal covariance of the \( X \) variables for both classes of observations, but subsequent iterations consider the two separate estimated covariance matrices for the two groups.

We have estimated quadratic and iterative linear discriminant functions for classification of the data of section 3.3 into "rescheduling" and "non-rescheduling" country-years. Only the ratio of debt service to exports
(X_1) and the amortization/debt ratio (X_6) were used in these estimates, since only these two variables were significant in the regression approach (aside from the imports/reserves variable which is difficult to predict).

In the iterative linear approach, the tenth iteration was used. The parameter estimates for these models are shown in Table VI. In model D, variables X_1 and X_6 were used. In model E, X_6 was replaced by its reciprocal, the ratio of debt to amortization, so that increasing debt service difficulty would be indicated by increasing values of both X-variables. Since this variable takes on some very large values (over 100, despite a median value of about 12), model F was also estimated, in which the natural logarithm of debt/amortization replaces the X_6 variable.

The quadratic, simple linear, and iterated linear discriminant functions were applied to the set of past data to predict "rescheduling" of debt. Versions D, E, and F were applied to each of these three approaches. Furthermore, alternative critical values were used for each model. For the quadratic models, one set of predictions was based on a critical value Z* derived on the assumption of equal a priori probabilities q_1 and q_2 of drawing from rescheduling and non-rescheduling "populations" and equal costs of misclassification, C(2|1) and C(1|2), while another was based on a critical value Z* in which the a priori probabilities were estimated by q_1 = k_1/(k_1 + k_2) and q_2 = k_2/(k_1 + k_2) where k and k_2 are the number of observations in the scheduling and non-rescheduling groups. These critical values are denoted by Z*_Q1 and Z*_Q2, respectively, in Table VII. For the linear models, three alternative critical values were used:

(i) the simple average of the means of the discriminant function is scheduling and non-rescheduling groups;
Table VI
Quadratic, Linear, and Iterative Linear Discriminant Functions

Quadratic Model

\[
D \quad Z = 35.606x_1^2 - 342.78x_1x_6 - 54.369x_6^2 + 42.122x_1 + 74.503x_6 - 9.33
\]

\[
E \quad Z = -163.055x_1^2 + 0.9182x_1x_6 - 0.002x_6^2 - 74.503x_1 + 0.3638x_6 - 7.669
\]

\[
F \quad Z = -6.081x_1^2 + 29.102x_1x_6 - 0.3438x_6^2 - 46.146x_1 - 4.426x_6 + 8.476
\]

Linear Model

D (1st iteration) \quad Z = 27.285x_1 - 20.380x_6

(10th iteration) \quad Z = 50.844x_1 - 45.825x_6

E (1st iteration) \quad Z = 22.298x_1 + 0.020x_6

(10th iteration) \quad Z = 40.719x_1 + 0.062x_6

F (1st iteration) \quad Z = 24.577x_1 + 0.949x_6

(10th iteration) \quad Z = 45.850x_1 + 2.500x_6
(ii) the average of the means in the two groups, each weighted inversely by the estimated standard deviation of the discriminant function in each group; and

(iii) the critical value in (ii) above plus the term \( \log \left( k_1/k_2 \right) \) to adjust for unequal estimated \textit{a priori} probabilities.

These three critical values are denoted by \( Z_{L1}^*, Z_{L2}^*, \) and \( Z_{L3}^* \) respectively in Table VII which gives the errors of prediction for all models and all critical values.

In terms of predictive power, the best model shown in Table VII is version F of the iterative linear model with the critical value based on the relative standard deviation of \( Z \) in the/estimated on/\textit{a priori} probabilities of rescheduling. Therefore this model is used for the projections of section 4.4. The model gave only 13 incorrect predictions for past data, i.e., correct predictions are made for 91 per cent of the observations. Its errors were relatively balanced between the truly rescheduling observations (5 errors in 13 observations) and the truly non-rescheduling observations (8 errors in 123 observations). Even so, the percentage frequency of error was greater for truly rescheduling observations than for truly non-rescheduling. This fact is important to remember when using this model for predictions; i.e., it tends to understate the number of rescheduling situations and overstate the number of non-rescheduling situations.

A final note concerning the prediction results is that the number and allocation of prediction errors is highly sensitive to the critical value chosen. In the linear models, for example, the errors are more sensitive to/critical value than to differences between the 10th iteration and the first
Table VII

Predictions Results: Quadratic, Linear, and Iterative Linear Discriminant

FUNCTIONS

<table>
<thead>
<tr>
<th>Model</th>
<th>Quadratic</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Z^*_Q1$</td>
<td>$Z^*_Q2$</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type I</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Type II</td>
<td>21</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>21</td>
<td>11</td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type I</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>Type II</td>
<td>18</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>20</td>
<td>12</td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type I</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>Type II</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>25</td>
<td>12</td>
</tr>
</tbody>
</table>
iteration of the model. Also, in the quadratic and linear models, the total number of errors declines when the \textit{a priori} probability information is incorporated into the critical value. While this procedure improves explanation for past data, it probably introduces a downward bias when used for projection of future reschedulings, since it is likely that the frequency of debt rescheduling in the future will be greater than it was over the nine-year period studied and therefore that the \textit{a priori} probabilities will shift toward a higher $q_1$ and lower $q_2$.

To summarize, when the variance of $X$-variables is different for the two classes of observations, the quadratic and iterative linear discriminant functions developed in the Appendix can improve prediction over that of the simple linear discriminant function. In the case of the data examined here, this expected improvement is more pronounced for the iterative linear model than for the quadratic model.

IV. Future Prospects

4.1 Existing Debt Service Projections

The total international debt of the less developed countries of the world was about $45 billion in 1966. The service on that debt amounted to about $5 billion.\footnote{United Nations Conference on Trade and Development (UNCTAD) Secretariat, \textit{The Outlook for Debt Service}, TD/7/ Supp. 5, mimeo, October 31, 1967, p. 2.} A number of projections of debt service have been made by various organizations.\footnote{Projections of debt service have been made by a number of national and international organizations including the United States State Department, the Development Assistance Committee (DAC) of the Organization for Economic Cooperation and Development (OECD), the World Bank (IBRD), and the UNCTAD Secretariat. A partial list includes the following: (cont'd on next page).} All seem to indicate an alarming growth in interest and amortization payments. For example, the Secretariat of the United Nations Conference on Trade and Development recently made two sets of projections for all less developed countries, one assumes that net capital
flows (gross capital flows less repayments) will remain at the 1966 level, at about 3 1/2 per cent per annum. The second set of projections, assuming a constant level of gross lending reveal that the reduction in net flows, due to an increasing repayment stream, will decrease by about 40 per cent by 1975.1/

4.2 Individual Projections for 17 Countries

While the existing debt projections seem to indicate increasing debt servicing difficulties for less developed countries, they are not very useful for the purpose of applying the composite indices computed above to distinguish between potential rescheduling and non-rescheduling countries. First, many of the projections are on a global basis and do not reveal individual country differences. Second, the few individual country


2 (Cont’d. from previous page)

(a) OECD, India’s Long-Run Foreign Debt Burden and the Terms of Assistance, DAC/FA (67)8, May 9, 1967.

(b) OECD, Appropriate Terms of Assistance — Pakistan Case Study, DAC/FA (67)1, January 17, 1967.

(c) OECD, Projections of Indonesia’s Debt Situation, DD-6, June 15, 1967.

(d) OECD, A Study of Turkey’s Long-Run Foreign Debt Position, Consortium, (Turkey /67/ 24, November 30, 1967, Parts I and II.

(e) I3RD, (by B. de Vries), The Outlook for Development Debt, November 13, 1967.


(g) UNCTAD, The Outlook for Debt Service, TD/7/Supp. 5, October 31, 1967.

(h) OECD, The Growth of External Debt and Debt Service of Developing Countries, DAC/FA(65)13, June 9, 1965 (revised version of DAC/FA(65)7.)
projections available are not made on a consistent basis, differing greatly in methodology and assumptions concerning growth in lending, trends in lending terms, and kinds of foreign debt included.

In order to provide a set of consistent estimates for a reasonably large number of countries, we made a set of projections based on 1967 data to the year 1977 for 17 countries. The 17 countries are:

Argentina Peru Bolivia Mexico
Turkey Korea Dominican Republic Indonesia
Chile Iran India
Colombia Nigeria Pakistan
Israel Tunisia Brazil

Fifteen of these countries rank in the top 20 in terms of total foreign debt outstanding. Together they have accounted for well over one-half of total foreign assistance received in the last decade. Many of them have already experienced debt servicing difficulties. Note that Ghana and the United Arab Republic, although heavily involved in the payment of debt service, are not included because of lack of data.

The foundation of the projections was, in most cases, IBRD estimates of service payments due during the period 1967-1992 on the basis of debt already outstanding at the beginning of 1967. Upon this foundation, we assumed new loan disbursements to these countries to occur at the same gross rate (or as an alternative, at the same net rate) as it had in the recent past (in most cases the last two or three years). The new lending was broken down into several term categories (usually from 4 to 7 categories) based on recent experience in borrowing by source. The

---

1 The IBRD data includes all debt which is payable to creditors outside the debtor country with an original maturity of one year or more which are obligations of governments or public agencies. However, it includes publicly guaranteed private debt. It does not include: (1) transactions with the International Monetary Fund; (2) non-guaranteed private debt; (3) local currency obligations; and (4) other minor categories of debt.
most recent set of loan terms of each lending sources, e.g., USAID
development loans or IDRC loans, were then applied to the appropriate
categories.

The resulting debt service projections are shown in Table IV.\(^1\) The
countries are grouped to preserve the confidentiality of the IDRC
data. Note that the bulk of the repayments over the ten-year period are
based on service due on debt already outstanding in 1967. Thus, while
the assumptions concerning the bulk and terms of new lending affect the
projections, the projections are not highly sensitive to these factors. Under
the constant gross aid assumption, countries in Group A will experience a
doubling of their debt service obligations in five years and a tripling
in ten years, a rate of growth of 11.1 per cent per annum over the ten-year
period. Under the constant net aid assumption, all 17 countries experience
substantial growth in debt service.

<table>
<thead>
<tr>
<th>Country Groups</th>
<th>Total Debt Service ($ million)</th>
<th>Index of Debt Service</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gross Aid Constant</td>
<td>Net Aid Constant</td>
</tr>
<tr>
<td></td>
<td>762.4</td>
<td>764.6</td>
</tr>
<tr>
<td>Group A</td>
<td>1563.5</td>
<td>1658.5</td>
</tr>
<tr>
<td></td>
<td>2313.3</td>
<td>2750.1</td>
</tr>
<tr>
<td></td>
<td>420.8</td>
<td>420.8</td>
</tr>
<tr>
<td>Group B</td>
<td>612.6</td>
<td>662.3</td>
</tr>
<tr>
<td></td>
<td>678.6</td>
<td>993.0</td>
</tr>
<tr>
<td></td>
<td>1556.3</td>
<td>1556.3</td>
</tr>
<tr>
<td>Group C</td>
<td>1228.0</td>
<td>1550.1</td>
</tr>
<tr>
<td></td>
<td>1344.3</td>
<td>2000.1</td>
</tr>
</tbody>
</table>

\(^1\) For a more detailed discussion of the basis of these projections see
Charles R. Frank, Jr., William R. Cline and Thomas Geweke, Debt Servicing
Problems of Less Developed Countries and Terms of Foreign Aid: With Special
Reference to United States Policy, mimeo, August 1968, Appendix C.
Group A countries which account for the bulk of debt service payments by 1977, experience nearly a quadrupling of debt service payments over ten years, a rate of growth of 13.7 per cent per annum.

One very interesting implication of the constant gross aid projections is that by 1977, ten of the seventeen countries reach a situation in which net capital flows reverse direction; that is payments of interest and amortization exceed the value of new lending. If the projections are extended to 1984, all 17 countries reach the turning point in net flows of foreign assistance.

4.3 Debt Service Ratios

In order to translate the debt service projections into indicators of debt servicing difficulty, we first computed the implied debt service ratios. These ratios were based on three alternative assumptions about export growth:

(i) a continuation of the 1960-1967 export growth trend for each individual country,

(ii) a four per cent rate of export growth, and

(iii) an eight per cent rate of export growth.

The countries are grouped into three categories, I, II, and III, representing high, medium, and low growth of debt service and levels of debt service in 1977, respectively. The projected debt service ratios for the group II countries tend to be much higher than those experienced by any but a few countries in modern history, exceeding in some cases more than 50 per cent of export earnings by 1977. The debt service ratios for the middle group II are also alarmingly high when viewed in historical perspective.
Table V
Projected Debt Service Ratios

<table>
<thead>
<tr>
<th>Country Group</th>
<th>Constant Gross Aid</th>
<th>Constant Net Aid</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Export Growth Rate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1960-67 Trend</td>
<td>4 per cent 8 per cent</td>
<td></td>
</tr>
<tr>
<td>III (three countries)</td>
<td>.052 .053 .062</td>
<td>.052 .054 .083</td>
<td>.052 .062 .067</td>
</tr>
</tbody>
</table>

4.4. Projecting the Discriminant Function

To further examine the likelihood of future debt servicing difficulties, we applied the discriminant functions estimated in section 3.4 to projected debt service/export and amortization/debt ratios. The projections of the debt service ratio are those described in the previous section and the projections of the amortization rate were based on the projections of debt service discussed in section 4.2. Table VI gives the years in which the iterated linear discriminant function (10th iteration) exceeded its critical value. The critical value was determined as a weighted average of discriminant function means in scheduling and non-rescheduling country groups plus an adjustment for estimated a priori probabilities for a country rescheduling. The weights were the estimated standard deviations of the discriminant functions in the two groups (see
Table VI
Projections of Discriminant Function
Years for Which Critical Value Is Exceeded
(Numbers in parentheses indicate number of years for which critical value is exceeded)

<table>
<thead>
<tr>
<th>Gross Aid Constant Export Growth:</th>
<th>Net Aid Constant Export Growth:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960-67 4 per cent</td>
<td>1960-67 4 per cent</td>
</tr>
<tr>
<td>Trend 8 per cent</td>
<td>Trend 8 per cent</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Group I</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 India</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'88</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(26)</td>
<td>(26)</td>
<td>(22)</td>
<td>(26)</td>
<td>(26)</td>
<td>(26)</td>
<td></td>
</tr>
<tr>
<td>2 Pakistan</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'89</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'92</td>
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</tr>
<tr>
<td></td>
<td>(26)</td>
<td>(26)</td>
<td>(23)</td>
<td>(26)</td>
<td>(26)</td>
<td>(26)</td>
<td></td>
</tr>
<tr>
<td>3 Indonesia</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'78; '82</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'84</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(26)</td>
<td>(26)</td>
<td>(13)</td>
<td>(26)</td>
<td>(26)</td>
<td>(18)</td>
<td></td>
</tr>
<tr>
<td>4 Tunisia</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'78</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'90</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(26)</td>
<td>(26)</td>
<td>(12)</td>
<td>(26)</td>
<td>(26)</td>
<td>(24)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Group II</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5 Brazil</td>
<td>'67-'68</td>
<td>'67-'68</td>
<td>'67-'68</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'78</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td>(2)</td>
<td>(2)</td>
<td>(26)</td>
<td>(26)</td>
<td>(12)</td>
<td></td>
</tr>
<tr>
<td>6 Mexico</td>
<td>'67-'79</td>
<td>'67-'86</td>
<td>'67-'76</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'76</td>
<td></td>
</tr>
<tr>
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<td>(13)</td>
<td>(20)</td>
<td>(10)</td>
<td>(26)</td>
<td>(26)</td>
<td>(10)</td>
<td></td>
</tr>
<tr>
<td>7 Argentina</td>
<td>'67-'69</td>
<td>'67-'69</td>
<td>'67-'68</td>
<td>'67-'92</td>
<td>'67-'92</td>
<td>'67-'84</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>(3)</td>
<td>(2)</td>
<td>(26)</td>
<td>(26)</td>
<td>(18)</td>
<td></td>
</tr>
<tr>
<td>8 Turkey</td>
<td>'68-'71</td>
<td>'69-'87</td>
<td>'68-'72</td>
<td>'68-'70</td>
<td>'68-'37</td>
<td>'68-'71</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>(20)</td>
<td>(5)</td>
<td>(3)</td>
<td>(20)</td>
<td>(4)</td>
<td></td>
</tr>
<tr>
<td>9 Chile</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>'75-'92</td>
<td>-</td>
<td></td>
</tr>
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<td>(0)</td>
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<td>(0)</td>
<td>(0)</td>
<td>(18)</td>
<td>(0)</td>
<td></td>
</tr>
<tr>
<td>10 Colombia</td>
<td>'74-'92</td>
<td>-</td>
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*Projections for '68-'87 only*
section 3.4).  

Table VI uses the same grouping of countries as in Table V. Countries in group I have values of the discriminant function exceeding the critical value for a considerable number of years. Prospects for successful debt servicing are exceedingly grim for countries in category I. These countries account for a very large portion of total foreign aid received; debt outstanding for these countries is per cent of projected total debt outstanding for the 17 countries in 1977.

Countries in group II generally have excessive values of the discriminant function if export growth is unfavorable and/or attempts are made to continue present levels of net lending. There is one exception, country 6. This country has a discriminant value which exceeds the critical value for a large number of years regardless of the export growth or aid assumption. The discriminant value exceeds the critical value by only a small amount in every case, and this particular country has managed to avoid default despite a high debt service ratio in the past.

Countries in group III should be relatively free of debt servicing problems for the next decade or so at least unless, however, particularly unfavorable developments occur.

---

1The other discriminant functions and critical values discussed in section 3.4 were also used for projections. The results, however, were very similar to those presented in Table VI.
4.5 An Overall Assessment of Prospects

For years in which the discriminant value exceeds the critical level, there is some presumption that countries will be highly likely to feel that the burden of debt service will be so heavy that they will request some form of debt relief from their creditors. This assumes that roughly the same factors which operated on rescheduling countries in the past are likely to operate in the future. There may, however, be mitigating circumstances in the future that will make the debt service burden less onerous. First, an increasing number of foreign aid recipients are serviced by consortia of aid donors who are likely to take into account the debt service burden when setting aid levels and policies. Secondly, the buildup of debt service is relatively easy to foresee given the availability of data on government and government guaranteed loans outstanding. Appropriate policies for adjusting to the increased level of debt service can be mitigated well ahead of time. In many past rescheduling exercises, debt servicing difficulties arose suddenly and without warning as the result of excessive reliance on short-term, non-guaranteed export credits for which little data were available.

On the other side, however, there are many reasons to believe that debt servicing difficulties will even be more severe than indicated by our projections. First, both the constant gross aid and constant net aid assumptions used in the debt service projections are conservative in light of past experience and in view of reasonable estimates of LDC need for foreign capital. For example, between 1960-62 and 1965-67 gross aid from members of the Development Assistance Committee to the LDC's and multilateral agencies grew at 4.2 per cent annually and aid net of amortization and interest grew at 2.5 per cent.
Secondly, the debt service projections assume that terms of foreign lending will be roughly the same in the future as in the 1964-66 period. Several recent developments will very likely invalidate that assumption and cause a greater burden of debt service. United States terms on development loans have increased substantially since 1964.\footnote{In 1964, these terms were 1.0 per cent interest during a ten year grace period for repayment of principal and 2.5 per cent over the remaining life of the loan. In 1968 these two rates were raised to 2.0 and 3.0 per cent, respectively.} PL 480 assistance is gradually being shifted to a hard currency repayable basis by 1971 with terms similar to development loans. Another development is a prospective rapid increase in lending by countries (e.g. Germany and Japan), multilateral agencies (e.g. the World Bank, and the Inter-American Development Bank) and other institutions (e.g. the U.S. Export-Import Bank) which lend on commercial terms. Revised and more recent data which was just becoming available at the time this paper was written indicate that for a number of countries in groups I and II, the debt service projections for 1967 and 1968 are serious under estimates because of the hardening of average terms.

Thirdly, the data and projections for debt service exclude non-guaranteed export credits which can add significantly to the debt service burden and cause "lumpiness" in debt service payments which often has resulted in attempts to seek debt relief.

Finally, the projections of the discriminant function were done on a very conservative basis. The critical value was adjusted upwards to account for a priori probability of rescheduling estimated from past data.
Yet the projections themselves suggest much higher frequency of future rescheduling. The critical value is therefore probably overstated, and the number of projected reschedulings is probably understated. Furthermore, the particular model chosen for projection had low total errors but relatively high frequency of errors for truly rescheduling cases for past data.

On balance, then, the prospects for serious debt servicing difficulties implicit in the analysis in Table VI error on the optimistic rather than the alarmist side. Furthermore, it is significant to note that many of the countries in groups I and II have a debt structure heavily biased toward long term, low interest loans, notwithstanding their high level of debt service. This suggests that rescheduling exercises will be frequent and repetitive since it is extremely difficult for these countries to work themselves out of a high debt service situation by a temporary restriction on commercial borrowing.

V. Conclusions

The above analysis and projections suggest a very strong case for a substantial softening of the terms of foreign assistance. It also indicates a high degree of caution should be exercised with regard to any expended use of institutions which because of their use of borrowed capital must lend on commercial terms. Furthermore, special assistance efforts are necessary for those countries which are highly likely to have serious debt servicing problems. Either these countries must be assured of aid levels which will compensate for their high level of debt service or of access to some established procedures for rescheduling their debt payments. Denieć such assurances, default on international debt will
become increasingly attractive for these countries.

The analysis also suggests that export growth rates have a heavy influence on debt servicing capability. Repayment of foreign debt is possible only to the extent that eventually exports of goods and services exceed imports. While it is beyond the scope of this paper to consider trade policies in general, it is well to note that increased access to markets in developed countries and less tying of aid to purchases in developed countries are policies which can add to the ability of the LDC's eventually to transfer the real resources implicit in the commitment to repay foreign loans.
Mathematical

Appendix

Discriminant Analysis When Variances Are Unequal
I. Introduction

In analyzing economic data, one sometimes wishes to formulate a statistical model in which the dependent variable $y$ is not continuous. Thus one may wish to define a dependent variable which takes on integer variables. This frequently arises in either/or situations ($y = 0$ or $y = 1$). For example, a plant is built in location $A$ or it is not built; a country devalues or it does not devalue; a consumer becomes a homeowner or he does not.

There are a number of statistical models for handling this kind of situation. Linear regression analysis is the least satisfactory since it assumes that the dependent variable is composed of a systematic component and error component with certain properties. A zero-one variable usually cannot be thought of as satisfying these properties (see Goldberger [3], pp. 248-251). Tobin ([5] and [6]) has suggested the use of probit analysis which involves difficult computational problems. Ladd [4] proposes the use of discriminant analysis which is computationally equivalent to linear regression analysis although the coefficient of determination $R^2$ and the various $F$ and $t$ statistics associated with regression analysis
must be interpreted differently.\footnote{Ladd [4] is somewhat misleading on this point as he implies that since linear regression and discriminant analysis are equivalent computationally, the same tests of significance hold. The sampling properties of the estimators, however, are not the same. See Anderson [2].}

In discriminant analysis, the independent variables
\[ x_1, x_2, \ldots, x_n \]
are assumed to be randomly distributed. The problem may be stated as to find a linear combination,

\[ z = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \tag{1} \]

of the random independent variables and a critical value \( z^* \) of \( z \) such that if \( z \geq z^* \), the dependent variable \( y \) is assumed to be equal to 0 and if \( z < z^* \), \( y \) is assumed equal to unity. That is, for a given column vector \( x = (x_1, x_2, \ldots, x_n) \)

\[ y = \begin{cases} 0 & \text{if } z = \beta \cdot x \geq z^* \\ 1 & \text{if } z = \beta \cdot x < z^* \end{cases} \]

where \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) is a row vector.

Another way of looking at the problem is to find a hyperplane \( z^* = \beta \cdot x \) in \( n \)-dimensional space which partitions that space into two regions I and II (see Fig. 1) such that if \( x \) lies in region I, \( y = 0 \), and if \( x \) lies in II, then \( y = 1 \).

(Figure 1)
One basic assumption in discriminant analysis, however, is that the random variable $x$ is assumed to have the same variance whether $y = 0$ or $y = 1$. Clearly this may not be a proper assumption in practical applications, e.g., the variance of incomes among automobile owning households may not be the same as the variance of incomes among non-owning households. The purpose of this paper is to analyze various discriminant functions and estimating procedures when variances are unequal in the $y = 0$ and $y = 1$ categories.

II. Bayes Procedures

Following Anderson [1] (pp. 126-153), we define regions of classification based on an analysis of conditional probabilities. A classification scheme based on these regions is called a Bayes procedure. Suppose there are two populations $\pi_1$ and $\pi_2$. $x$ is a vector of $n$ elements drawn from either $\pi_1$ or $\pi_2$. Given the observation $x = (x_1, \ldots, x_n)$ for which the population is not known, we wish to partition the $n$-dimensional Euclidean space into regions $R_1$ and $R_2$ such that if $x$ lies in $R_1$, we classify $x$ as having come from $\pi_1$ and if $x$ lies in $R_2$, we classify $x$ as having come from $\pi_2$. Thus in Figure 1, the region $R_1$ is defined as the set of points $x = (x_1, x_2)$ such that $\beta_1 x_1 + \beta_2 x_2 \geq z^*$. In Figure 2, the region $R_1$ is the interior of an ellipse, i.e., the set of points $x = (x_1, x_2)$ such that $\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_1 x_2 + \beta_4 x_1 + \beta_5 x_2 > z^*$.

Two methods have been proposed for choosing $R_1$ and $R_2$. (See Anderson [1]). If the probabilities $q_1$ and $q_2$ of drawing
an observed \( x \) from \( \pi_1 \) and \( \pi_2 \), respectively, are known, then a Bayes procedure is one in which \( R_1 \) and \( R_2 \) are chosen to minimize the expected cost of a classification error:

\[
(2) \quad c = q_1 \cdot c(2|1) \cdot P(\pi_1 | x \in R_2) + q_2 \cdot c(1|2) \cdot P(\pi_2 | x \in R_1),
\]

where \( c(2|1) \) is the cost of classifying an observation in \( \pi_2 \) when it comes from \( \pi_1 \) and \( P(\pi_1 | x \in R_2) \) is the probability that \( x \) comes from \( \pi_1 \) given that \( x \) lies in \( R_2 \) (Type I error). \( c(1|2) \) and \( P(\pi_2 | x \in R_1) \) are similarly defined and \( P(\pi_2 | x \in R_1) \) is called the probability of a Type II error.\(^1\)

(Figure 2)

When \( q_1 \) and \( q_2 \) are not known, one may use a minimax procedure, i.e., choose \( R_1 \) and \( R_2 \) so that we

\[
(3) \quad \text{minimize } \max_{R_1, R_2} \{ c(2|1) \cdot P(\pi_1 | x \in R_2), c(1|2) \cdot P(\pi_2 | x \in R_1) \}.
\]

One can show that a minimax procedure is also a Bayes procedure for some \( q_1 \) and \( q_2 \). (See Anderson [1] p. 147.) Thus let us first analyze the case of known probabilities.

---

\(^1\) These definitions of Type I and Type II errors are not the same as the customary definition of Type I and Type II errors in the theory of tests of statistical hypotheses.
Let $p_1(x)$ be the probability density function of population $\pi_1$ and $p_2(x)$ be the probability density function of $\pi_2$. Note that the expression (2) for expected cost may be interpreted as if the probability of drawing from $\pi_1$ is $\pi_1^* = p_1 \cdot q(2|1)$ and from $\pi_2$ is $\pi_2^* = p_2 \cdot q(1|2)$ and that the costs are equal to unity. Then the conditional probability that a given $x$ comes from $\pi_1$ is

$$\Pr(\pi_1|x) = \frac{\pi_1^* \cdot p_1(x)}{\pi_1^* \cdot p_1(x) + \pi_2^* \cdot p_2(x)}.$$  \hspace{1cm} (4)

Similarly,

$$\Pr(\pi_2|x) = \frac{\pi_2^* \cdot p_2(x)}{\pi_1^* \cdot p_1(x) + \pi_2^* \cdot p_2(x)}.$$  \hspace{1cm} (5)

For each given $x$, we wish to choose $R_1$ so that the probability that $x$ comes from $\pi_1$ is greater than the probability that $x$ comes from $\pi_2$.

---

1. The probabilities $\Pr(\pi_1|x)$ and $\Pr(\pi_2|x)$ in (4) and (5) may be derived as follows: Let $\Pr(\pi_i|x^0 \leq x \leq x^2)$ be the probability that the observation $x$ comes from $\pi_i$ given that it lies in the closed interval $I = (x^0, x^2)$ where $x^0$ and $x^2$ are $n$-dimensional vectors such that $x_i > x_i^0$ for all $i$. Then, using the definition of conditional probabilities,

$$\Pr(\pi_i|x^0 \leq x \leq x^2) = \frac{\pi_i^* \cdot \int_{x^0}^{x_2} p_i(x)dx}{\int_{x^0}^{x_2} p_1(x)dx + \pi_2^* \cdot \int_{x^0}^{x_2} p_2(x)dx}$$

Taking limits of both sides of this equality as $x_i \to x_i^0$ where $x_i^0$ is held fixed, we obtain the relationship (4). L'Hopital's rule may be used to take the limit of the ratio on the right hand side. The probability in (5) is derived in a similar fashion.
Thus

\[(6) \quad P(\pi_1 | x) \geq P(\pi_2 | x) .\]

From (1), (5), and (6), we see that

\[(7) \quad R_2 = \{x | p_1(x)/p_2(x) \geq \frac{q_2 \cdot c(1|2)}{q_1 \cdot c(2|1)}\} .\]

The region \(R_2\) is, of course, the set of points not belonging to \(R_1\).

III. Quadratic Discriminant Functions

Suppose \(x_1\) and \(x_2\) have multivariate normal distributions \(N(\mu_1, \Sigma_1)\) and \(N(\mu_2, \Sigma_2)\), respectively, where \(\mu_1 = (\mu_{11}, \mu_{12}, \ldots, \mu_{1n})\) and \(\mu_2 = (\mu_{21}, \mu_{22}, \ldots, \mu_{2n})\) are \(n\)-dimensional column vectors and \(\Sigma_1 = (\sigma_{1ij}^2)\) and \(\Sigma_2 = (\sigma_{2ij}^2)\) are \(n \times n\) covariance matrices. Let

\[(8) \quad k = \frac{q_2 \cdot c(1|2)}{q_1 \cdot c(2|1)} .\]

The region \(R_1\) may be defined by determining the set of \(x\) for which \(\frac{p_1(x)}{p_2(x)} \geq k\). Since both sides of this inequality are positive, we can work with the inequality.

\[(9) \quad \log p_1(x) - \log p_2(x) \geq \log k .\]
which under the normal distribution assumption becomes

\begin{equation}
(10) \quad -\frac{1}{2} \log |\Sigma_1| + \frac{1}{2} \log |\Sigma_2| - \frac{1}{2}(x-\mu_1)' \Sigma_1^{-1} (x-\mu_1)
+ \frac{1}{2} (x-\mu_2)' \Sigma_2^{-1} (x-\mu_2) \geq \log k
\end{equation}

The left-hand side of (10) is the sum of two quadratic forms in \( x \),
a linear function of \( x \) and a constant. The region \( B_1 \) as defined by
the inequality (10) provides a Bayes procedure classification scheme.

Let us rewrite (10) as follows:

\begin{equation}
(11) \quad z = x'Ax + \beta'x + \zeta > z^* ,
\end{equation}

where \( z \) is the quadratic discriminant function, \( z^* = \log k \) is the
critical value, \( A = \Sigma_2^{-1} - \Sigma_1^{-1} \) is an \( nxn \) matrix,
\( B = \mu_2 - \mu_1 \) is an \( nx1 \) vector and
\( \zeta = \frac{1}{2} \mu_1' \Sigma_1^{-1} \mu_1 - \frac{1}{2} \mu_2' \Sigma_1^{-1} \mu_2 \) is a constant. If the two
covariance matrices \( \Sigma_2 \) and \( \Sigma_1 \) are equal to \( \Sigma \), the discriminant
function in (11) reduces to

\begin{equation}
(12) \quad z = \beta'x + \zeta > z^* ,
\end{equation}

where \( \beta = (\mu_1 - \mu_2)' \Sigma^{-1} \)
and \( \zeta = -\frac{1}{2}(\mu_1 + \mu_2)' \Sigma^{-1} (\mu_1 - \mu_2) \)
is a constant.

In the linear case (12) we have the regions of discrimination
as shown in Fig. 1. In the more general case (11), however, the
discriminant function is a quadratic form such as the ellipse shown in Fig. 2. The relative variances of the two populations are represented in Fig. 2 by the diameters of the dotted circles around the two means. As the variance of population $\pi_2$ gets larger relative to that of $\pi_1$, the discriminating ellipse becomes smaller and smaller, degenerating in the limit to a single point $(\mu_1, \mu_2)$.

Up to this point we have assumed that $\Sigma_1, \Sigma_2, \mu_1,$ and $\mu_2$ are known. If they are not known but $k_1$ sample observations from $\pi_1$ are

\[
(x_{1t} = (x_{11t}, x_{12t}, \ldots, x_{1nt}) \text{ for } t = 1, \ldots, k_1)
\]

and from $\pi_2$ are

\[
(x_{2t} = (x_{21t'}, x_{22t'}, \ldots, x_{2nt'}) \text{ for } t = 1, \ldots, k_2)
\]

available, the maximum likelihood estimates of these parameters are

\[
\hat{\mu}_1 = \bar{x}^1 = (\frac{\sum_{t=1}^{k_1} x_{11t'}, \ldots, \sum_{t=1}^{k_1} x_{1nt'}}{k_1})
\]

\[
\hat{\mu}_2 = \bar{x}^2 = (\frac{\sum_{t=1}^{k_2} x_{21t'}, \ldots, \sum_{t=1}^{k_2} x_{2nt'}}{k_2})
\]

(13)

\[
\hat{\Sigma}_1 = (\hat{\sigma}_{i,j}^2) = (\frac{\sum_{t=1}^{k_1} x_{1it} x_{1jt} - \bar{x}_i^1 \bar{x}_j^1}{k_1}) , \text{ and}
\]

(14)

\[
\hat{\Sigma}_2 = (\hat{\sigma}_{i,j}^2) = (\frac{\sum_{t=1}^{k_2} x_{2it} x_{2jt} - \bar{x}_i^2 \bar{x}_j^2}{k_2})
\]
When these estimates are substituted into (10), the region $\mathbb{F}_1$ so defined does not necessarily minimize expected cost of misclassification. As Anderson 1 (p. 139) has shown, however, as $\alpha_1$ and $\alpha_2$ approach infinity, the expected cost approaches its minimum value with a probability of 1. Thus the estimated regions of classification satisfy a kind of consistency test.

IV. General Minimax Procedures

We have also assumed that the costs $c(2\mid 1)$ and $c(1\mid 2)$ and the probabilities $q_1$ and $q_2$ of drawing an $x$ from population $\pi_1$ or $\pi_2$ are known. When these are not known, a possible procedure is to assume that $c(2\mid 1) = c(1\mid 2)$ and that $q_1 = q_2$. Then

$$k = q_2 \cdot c(1\mid 2) / q_1 \cdot c(2\mid 1) = 1$$

and $\log k = 0$ in the inequality (10). In this case, the region $\mathbb{F}_1$ defined by (10) can be thought of as the region which minimizes the probability of Type I and Type II errors.

If $c(2\mid 1)$ and $c(1\mid 2)$ are known or assumed to be equal and $q_1$ and $q_2$ are not known, a minimax solution may be appropriate. The probability of a Type I error, $p(\pi_1 \mid x \in \mathbb{F}_1)$, may be determined by integrating the multivariate normal distribution with mean $\mu_1$ and variance $\Sigma_1$ over the region $\mathbb{R}_2$. The probability of a Type II error is determined by integrating $N(\mu_2, \Sigma_2)$ over $\mathbb{F}_1$. Since $\mathbb{F}_1$
decreases in size as \( k \) increases in size and \( R_2 \) increases, the probability of a Type I error increases and of a Type II error decreases as \( k \) increases. Thus a minimax solution occurs whenever \( k \) is chosen so that the expected costs of misclassification are equal:

\[
(16) \quad c(2|1) \cdot P(\pi_1 | x \in R_2) = c(1|2) \cdot P(\pi_2 | x \in R_1)
\]

The process of iterating on \( k \) to find the value which causes (16) to hold can be extremely tedious and difficult. In the case of linear discriminant functions, however, there are much simpler procedures.

V. Linear Discriminant Functions

In previous sections we have defined regions of classification based on Bayes procedures. We saw that in general the regions of classification were based on critical values of a quadratic discriminant function. In this section we confine our attention to the class of linear discriminant functions.

Let

\[
(17) \quad z = \beta \cdot x
\]

be a linear discriminant function and let the regions of classification be
\( R_1 = \{ x \mid \beta \cdot x \geq z^* \} \), and
\( R_2 = \{ x \mid \beta \cdot x < z^* \} \),

where \( z^* \) is a critical value of \( z \). We wish to choose coefficients \( \beta \) and a critical value \( z^* \) which produce a minimax solution, i.e., regions of classification which satisfy (3) which by our argument in the previous section is equivalent to (16). Let us assume that if \( x \) comes from one of two multivariate normal distributions, \( N(\mu_1, \Sigma_1) \) or \( N(\mu_2, \Sigma_2) \), depending on whether \( x \) comes from population \( \pi_1 \) or \( \pi_2 \), respectively, then \( z \) has the single-variate normal distribution \( N(\beta \cdot \mu_1, \beta \cdot \Sigma_1 \cdot \beta') \) or \( N(\beta \cdot \mu_2, \beta \cdot \Sigma_2 \cdot \beta') \) depending on whether \( x \) is from \( \pi_1 \) or \( \pi_2 \). Then (16) becomes

\[
|z| \int_{-\infty}^{z^*} g(z; \beta \cdot \mu_1, \beta \cdot \Sigma_1 \cdot \beta') \, dz = |z| \int_{-\infty}^{z^*} f(z; \beta \cdot \mu_2, \beta \cdot \Sigma_2 \cdot \beta') \, dz,
\]

where \( f(z; \mu, \sigma^2) \) is the normal distribution function with mean \( \mu \) and variance \( \sigma^2 \). In Fig. 3, the integrals on the left and right hand sides of (19) are shown as the left and right hand tails of the two normal distributions.

(Figure 3)

Let \( z^* \) be expressed as a linear combination of the means, \( \beta \cdot \mu_1 \) and \( \beta \cdot \mu_2 \) of \( \pi_1 \) and \( \pi_2 \), respectively, i.e.,
(20) \[ z^* = \lambda \cdot \beta \cdot \mu_1 + (1-\lambda) \cdot \beta \cdot \mu_2 \]

where \( \lambda \) is a scalar. Then if \( t \) is the standard normal variate, (17) can be rewritten in terms of standard normal distribution functions as

\[
\Phi(\frac{-(1-\lambda) \cdot \beta \cdot (\mu_1 - \mu_2)}{(\beta \cdot \Sigma \cdot \beta')^{1/2}}) + \Phi(\frac{-\lambda \cdot \beta \cdot (\mu_1 - \mu_2)}{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2}}),
\]

where we used the symmetric property of normal distributions, i.e.,

(22) \[ \int_{-\infty}^{a} f(t;0,1)dt = \int_{-a}^{\infty} f(t;0,1)dt \]

Let us assume that \( \Phi(2|1) = \Phi(1|2) \). Then note that the left and right hand sides of (21) are increasing functions of the upper limits of integration and that the left and right hand sides are equal if their limits of integration are equal. Thus to find the minimax solution we minimise one of the limits of integration subject to the constraint that the limits are equal. That is,

(23) \[ \max_{\lambda, \beta} \left[ \frac{\lambda \cdot \beta \cdot (\mu_1 - \mu_2)}{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2}} \right] \]
subject to

\[
(24) \quad \frac{\lambda \cdot \beta \cdot (\mu_1 - \mu_2)}{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2}} = \frac{(1 - \lambda) \cdot \beta \cdot (\mu_1 - \mu_2)}{(\beta \cdot \Sigma_1 \cdot \beta')^{1/2}},
\]

or

\[
(25) \quad \lambda = \frac{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2}}{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2} + (\beta \cdot \Sigma_1 \cdot \beta')^{1/2}}
\]

Substitute this value of \( \lambda \) into (23) to obtain

\[
(26) \quad \text{Maximize} \quad \beta \left[ \frac{\beta \cdot (\mu_1 - \mu_2)}{(\beta \cdot \Sigma_1 \cdot \beta')^{1/2} + (\beta \cdot \Sigma_2 \cdot \beta')^{1/2}} \right]
\]

Note that (26) is invariant with respect to scalar multiplication of the vector \( \beta \). Thus we may confine our attention to a normalized set of \( \beta \) coefficients satisfying

\[
(27) \quad \beta \cdot (\mu_1 - \mu_2) = c
\]

where \( c \) is a constant. In this case we minimize the denominator of (26) subject to (27). If \( \rho \) is a Lagrangian multiplier, we minimize
(28) \[(\beta \cdot \Sigma \cdot \beta')^{1/2} + (\beta \cdot \Sigma_2 \cdot \beta')^{1/2} - \rho \{ \beta \cdot (\mu_1 - \mu_2) - c \} \] .

If we differentiate (28), we obtain the following normal equations.

\[
\left\{ \frac{\Sigma_1}{(\beta \cdot \Sigma_1 \cdot \beta')^{1/2}} + \frac{\Sigma_2}{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2}} \right\} \cdot \beta' = \mu'(\mu_1 - \mu_2) \quad \text{and}
\]

(29) \[(\mu_1 - \mu_2)^t \cdot \beta' = c \] .

These normal equations are nonlinear functions of the unknown \( \beta' \)'s. Although \( c \) is an arbitrary constant and \( \mu' \), the Lagrangian multiplier a variable, it is useful to regard \( \mu \) as an arbitrary constant and solve for \( c \). Specifically, let

(30) \[\rho = \frac{(\beta \cdot \Sigma_1 \cdot \beta')^{1/2} + (\beta \cdot \Sigma_2 \cdot \beta')^{1/2}}{(\beta \cdot \Sigma_1 \cdot \beta')^{1/2} (\beta \cdot \Sigma_2 \cdot \beta')^{1/2}}\]

Then the first set of normal equations (29) becomes

(31) \[\left\{ \frac{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2} \cdot \Sigma_1 + (\beta \cdot \Sigma_1 \cdot \beta')^{1/2} \Sigma_2}{(\beta \cdot \Sigma_2 \cdot \beta')^{1/2} + (\beta \cdot \Sigma_1 \cdot \beta')^{1/2}} \right\} \cdot \beta' = (\mu_1 - \mu_2)\]
Note that the matrix on the left hand side of (31) is a weighted average of the two covariance matrices $\Sigma_1$ and $\Sigma_2$ where the weights are the standard deviations of the discriminant function $z = \beta \cdot x$ for populations $\pi_2$ and $\pi_1$, respectively.

The set of $\beta$ coefficients which satisfy (31) may be determined by means of the following iterative procedure: First, assume that the covariance matrices $\Sigma_1$ and $\Sigma_2$ are equal to $\Sigma$ and obtain a pooled estimate of $\Sigma$.

$$(32) \quad \hat{\Sigma} = \frac{k_1 \hat{\Sigma}_1 + k_2 \hat{\Sigma}_2}{(k_1 + k_2)}$$

where $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ are given by (12). Under the assumption of equal covariance matrices, (31) becomes

$$(33) \quad \Sigma \cdot \beta' = (\mu_1 - \mu_2) .$$

Solving for $\beta'$ and using estimates of $\Sigma$, $\mu_1$ and $\mu_2$, we obtain

$$(34) \quad \hat{\beta}' = \hat{\Sigma}^{-1} \cdot (\hat{\mu}_1 - \hat{\mu}_2) = \beta(1) .$$

Equation (34) gives us an initial estimate of $\beta$, namely $\beta(1)$. Given the $t^{th}$ estimate $\hat{\beta}(t)$, the $(t+1)^{st}$ approximation is obtained in the following manner. The $t^{th}$ estimate of the standard deviation of $z = \beta(t) \cdot x$ in populations $\pi_1$ and $\pi_2$, respectively, are
\[
\hat{\sigma}_{z_1}(t) = (\beta(t) \cdot \hat{\Sigma}_1 \cdot \beta'(t))^{1/2}, \text{ and} \\
\hat{\sigma}_{z_2}(t) = (\beta(t)^2 \cdot \hat{\Sigma}_2 \cdot \beta'(t))^{1/2}.
\]

Substitute these estimators into the matrix on the left hand side of (31) and solve for \( \beta' \) to obtain \( \beta(t+1) \).

\[
(36) \quad \beta'(t+1) = \left[ \frac{\hat{\sigma}_{z_2}(t)^2 \cdot \hat{\Sigma}_1 + \hat{\sigma}_{z_1}(t)^2 \cdot \hat{\Sigma}_2}{\hat{\sigma}_{z_1}^2 + \hat{\sigma}_{z_2}^2} \right]^{-1} \cdot (\hat{\mu}_1 - \hat{\mu}_2)
\]

If this process is terminated after \( T \) iterations, the critical value of \( Z \) is obtained by substituting \( \hat{\sigma}_{z_1}(T) \) and \( \hat{\sigma}_{z_2}(T) \) into (25) to obtain \( \lambda \) which is substituted (20) along with \( \beta(T) \) to obtain the critical value \( Z^* \).

There is no assurance that this iteration procedure converges to the true solution to (31). In all the practical examples and sample problems which have been run on the computer to date, however, the convergence is quite rapid.

There is reason to believe that the above procedure is not very efficient in the case when \( k_1 \) and \( k_2 \), the number of observations in each group are very different. The estimated covariance matrices \( \hat{\Sigma}_1 \) and \( \hat{\Sigma}_2 \) in (36) are weighted by the standard deviations of \( Z \) in each group. This does not take account of the fact that \( \hat{\Sigma}_1 \) and \( \hat{\Sigma}_2 \) may be of vastly differing quality in terms of the variance of the estimated covariances. Thus by combining the weights
in (36) and (32), we might achieve more efficiency. That is, estimate \( \beta(t+1) \) from

\[
(37) \quad \beta'(t+1) = \left[ \frac{k_1 \hat{\sigma}_1^2 \cdot \Sigma_1 + k_2 \hat{\sigma}_2^2 \cdot \Sigma_2}{(k_1+k_2)(\hat{\sigma}_1^2 + \hat{\sigma}_2^2)} \right]^{-1} \cdot (\hat{\mu}_1 - \hat{\mu}_2)
\]

VI. Concluding Remarks

Note that the numerator of the expression in (24) is proportional to the standard deviation between the two population means \( \beta \mu_1 \) and \( \beta \mu_2 \). The denominator is the sum of the standard deviations within the two populations. If \( \Sigma_1 = \Sigma_2 = \Sigma \), this expression becomes

\[
(38) \quad \frac{\beta \cdot (\mu_1 - \mu_2)}{2(\beta \cdot \Sigma \cdot \beta')^{1/2}}
\]

the square of which is proportional to the ratio of the variance between groups to the common variance within groups. The sampling properties of this quantity are discussed by Anderson [2].

Few sampling properties of the estimates of \( \beta \) are known; although for \( \Sigma_1 = \Sigma_2 = \Sigma \) it is easy to show that \( \hat{\beta}' = \Sigma^{-1}(\hat{\mu}_1 - \hat{\mu}_2) \) is consistent.
References


