DYNAMIC PORTFOLIO ANALYSIS AND ITS APPLICATION TO THE PROBLEM OF EXPORT DIVERSIFICATION

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I. Introduction

The classical approach to the analysis of portfolios is to choose a mix of portfolio assets (alternatively, a mix of exportable products) which maximizes a function of both expected returns and variance of returns. If, however, the returns exhibit either an autoregressive or lagged cross-covariance structure, this kind of analysis can be extremely misleading. The lag structure must be incorporated into the analysis to arrive at meaningful results.

Let us illustrate the point with a specific example. Suppose p_{1t} and p_{2t} are the average returns to assets (commodities) 1 and 2 at time t and that q_1 and q_2 are the amounts of each asset in the portfolio. The returns p_{1t} and p_{2t} are assumed to have equal variance σ^2. Suppose the planning or decision making period is two years, that is, a decision will be made to invest amounts q_1 and q_2 in each of the two assets in each of two years to maximize total expected returns for a given level of variance over the two years. If the returns to each commodity in year 1 are not affected by returns in year 2 and vice versa, and there is no covariance in returns from commodities 1 and 2, the total variance in returns can be written as

$$(1.1) \qquad V = 2\sigma^2(q_1^2 + q_2^2) \quad .$$

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For a given level of variance $\underline{V}$, the maximum level of expected revenue is given by the point $\underline{P}$ in Fig. 1. In particular, if $\underline{p}_{1t}$ and $\underline{p}_{2t}$ have the same expected value μ, the optimal policy is to set $\underline{q}_1 = \underline{q}_2 = \sqrt{\underline{V}}/2\sigma$.

Can this be good policy? Suppose the first price is highly positively autocorrelated and the second is highly negatively autocorrelated. Thus a portfolio heavily dependent on the first commodity will tend to have high returns in period 2 if returns are high in period 1 and low returns in period 2 if returns are low in period 1. If the portfolio is heavily weighted toward commodity 2, the opposite will be true. Low returns in period 1 will be associated with high returns in period 2. The variance of returns over the two periods will tend to be high with commodity 1 and low with commodity 2, although the commodities have the same variance in any single period.

In particular, let us assume that the returns $\underline{p}_{1t}$ and $\underline{p}_{2t}$ follow an autoregressive structure of the form:

$$(1.2) \qquad (\underline{p}_{1t}-\mu) = 0.9(\underline{p}_{1,\underline{t}-1}-\mu) + \underline{z}_{1t}$$

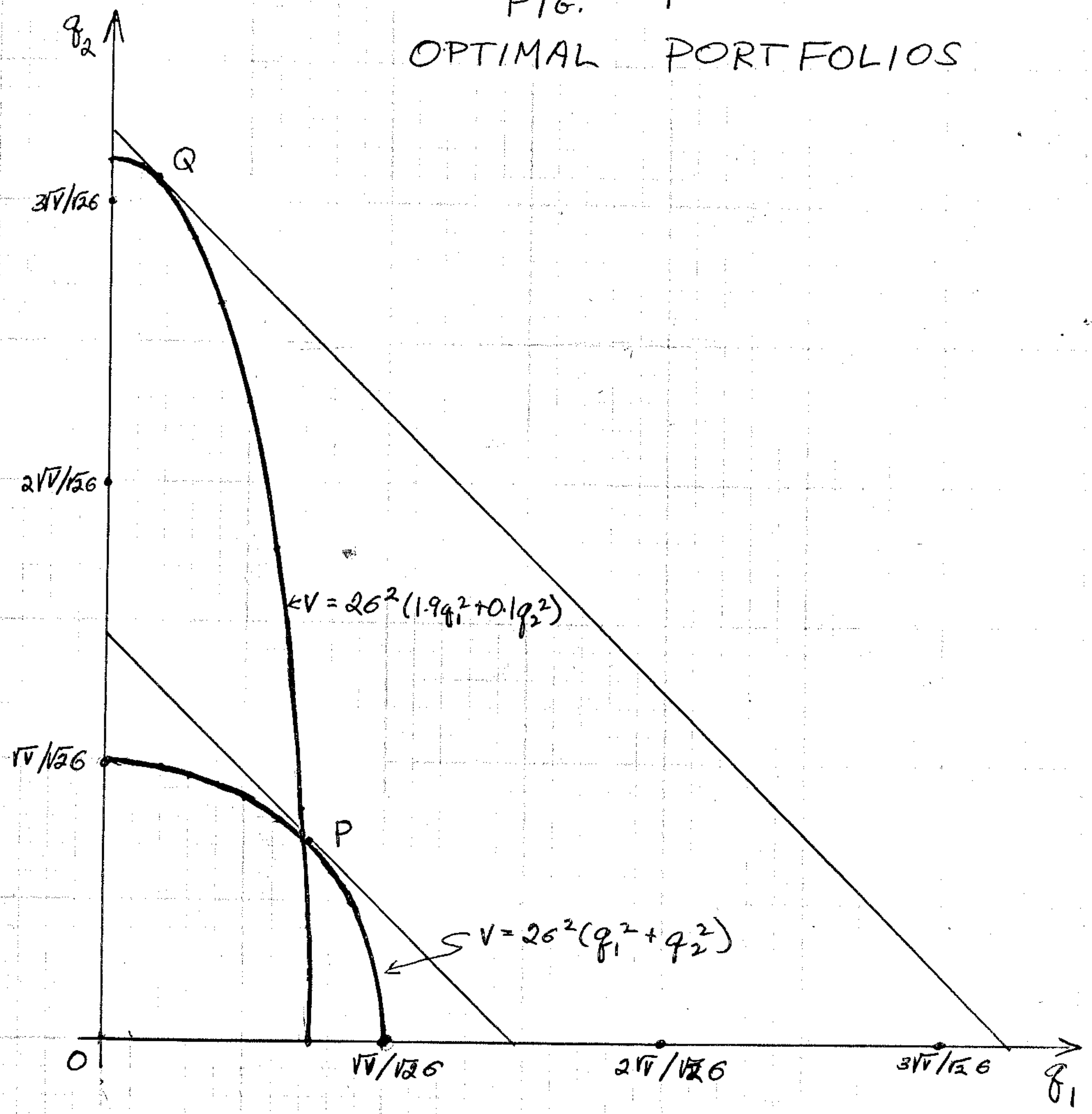
$$(1.3) \qquad (\underline{p}_{2t}-\mu) = 0.9(\underline{p}_{2,\underline{t}-1}-\mu) + \underline{z}_{2t}$$

where μ is the common mean of $\underline{p}_{1t}$ and $\underline{p}_{2t}$ and $\underline{z}_{1t}$ and $\underline{z}_{2t}$ are uncorrelated error terms, each with zero means, no autocorrelation, and constant variance σ_Z^2 $(0.19)\sigma^2$ through time. One can show that the variances of $\underline{p}_{1t}$ and $\underline{p}_{2t}$ both equal σ^2, as in the case above. The variance of revenue over the two period planning horizon, however, is given by*

* The expression (1.4) may be derived by noting that the correlation between the price p_{it} in period t and the price $p_{i,t+\tau}$ in period $t+\tau$ is given by $\gamma_{ii}(\tau) = \alpha_i^{\tau}\sigma_Z^2/(1-\alpha_i^2)$ where α_i is the first order autoregressive coefficient, 0.9 in (1.2) and -0.9 in (1.3).

FIG. 1

OPTIMAL PORTFOLIOS



(1.4) $$\mathrm{Var}(\underline{p}_{11}\underline{q}_1 + \underline{p}_{21}\underline{q}_2 + \underline{p}_{12}\underline{q}_1 + \underline{p}_{22}\underline{q}_2) = 2\sigma^2(1.9\underline{q}_1^2 + 0.1\underline{q}_2^2)$$

If the variance is constrained to a given level $\underline{V}$ and we maximize expected revenue, the optimal portfolio is given by the point $\underline{Q}$. Here $\underline{q}_2$ is given much more importance in the optimal portfolio since it is negatively autocorrelated and contributes to expected revenue to a greater degree for a given level of variance. Specifically, we have an optimal portfolio in which $\underline{q}_1 = 0.11\sqrt{\underline{V}}/\sigma$ and $\underline{q}_2 = 2.18\sqrt{\underline{V}}/\sigma$.* Note also that expected revenue is much higher when autocorrelation is taken into account. At the point $\underline{P}$, expected revenue is $2.0\sqrt{\underline{V}}/\sigma$ while at the point $\underline{Q}$, expected revenue is $4.6\sqrt{\underline{V}}/\sigma$, if we assume $\mu = 1.0$.

The importance of taking into account autocorrelation in portfolio analysis can be illustrated with reference to the problem of export diversification. Brainard and Cooper [1] have suggested that the portfolio model could be applied to export diversification. They computed the variance and covariance of the prices of a large number of commodities over the period 1951-1963. The variances and covariances were computed about the zero lag, i.e., no account was taken of serial correlation or cross correlation. They argued that, _ceteris paribus_, for commodity pairs with similar variability and high covariance, there is little reason to diversify the export mix toward one or the other of the pair since there would be little or no reduction in the variance of export earnings. (The _ceteris paribus_ here covers a multitude of other factors, including the assumption of no differences (on the margin) in net rates of return and no differences in the variance of uncontrollable fluctuations in output such as those caused by weather.)

* This solution can be determined graphically from Fig. 1.

held in the portfolio. We assume that enough of this riskless asset is held to cover a certain proportion of the variance in returns. For example, the riskless asset may be considered to be foreign exchange reserves which are held to cover shortfalls in foreign exchange earnings below the expected level of earnings.

2.1 The Dynamic Variance Function

Let $p'_t = (p_{1t},\ldots,p_{nt})$ be a vector of n stationary stochastic processes, say commodity prices, at time intervals $t = 1,\ldots,T$. Let $q' = (q_1,\ldots,q_N)$ be a vector of n quantity weights for the stochastic prices. For each lag interval τ ($\tau = 0, \pm 1, \pm 2,\ldots, \pm T$), define a covariance matrix

$$(2.1) \qquad \Gamma(\tau) = \begin{vmatrix} \nu_{11}(\tau) & \nu_{12}(\tau) & \cdots & \nu_{1n}(\tau) \\ \vdots & & & \\ \nu_{n1}(\tau) & \nu_{n2}(\tau) & \cdots & \nu_{nn}(\tau) \end{vmatrix} .$$

For example $\nu_{11}(5)$ is the autocovariance function of the first price process evaluated at the fifth lag; and $\nu_{12}(5)$ is the cross-covariance of the first and second prices, where p_2 leads p_1 by 5 time units. We note that

(1) $\Gamma(\tau)$ is symmetric for $\tau = 0$; also $\Gamma(0)$ is assumed positive definite;

(2) more generally, $\Gamma(\tau) = \Gamma'(-\tau)$ for every τ .

Then the export revenue accruing over T periods is the sum of vector products

$$(2.2) \qquad R_T = p_1 \cdot q + p_2 \cdot q + \ldots + p_T \cdot q \qquad .$$

The variance of this sum is

$$(2.3) \qquad \mathrm{Var}R_T = T \sum_{\tau=-T}^{T} (1-|\tau|/T)q'\Gamma(\tau)q$$

Note that in the static case where T = 1, (2.3) becomes

$$(2.4) \qquad \mathrm{Var}R_1 = q'\Gamma(0)q \quad .$$

That is, the static case involves the auto- and cross-covariances only at the zero lag.

The term $(1-|\tau|/T)$ in (2.3) will be recognized as Bartlett's lag window, a filter which is depicted in the time domain in Figure 2a. The filter accentuates autocovariances near the zero lag while autocovariances with long lags are attenuated.

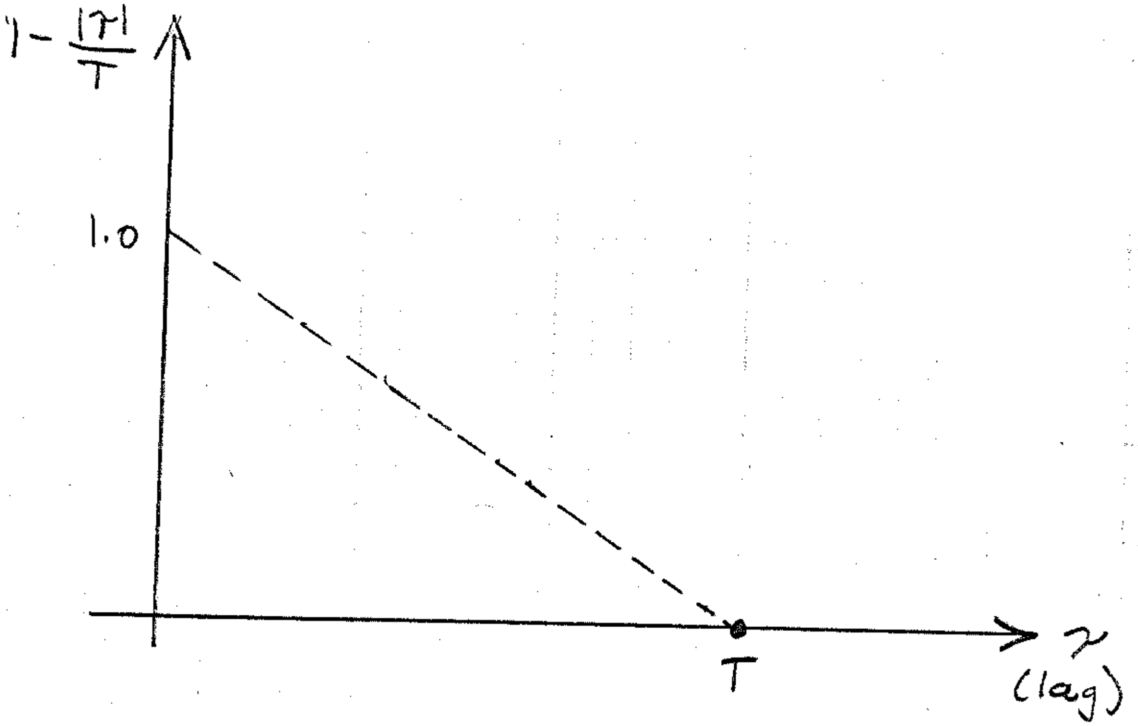
2.2 Frequency Interpretation

The frequency-domain interpretation of the Bartlett window is illuminating. Let $S_{jl}(f)$ be the power spectrum (if j = l) or the cross-spectrum (if $j \neq l$) in the Fourier transform identity

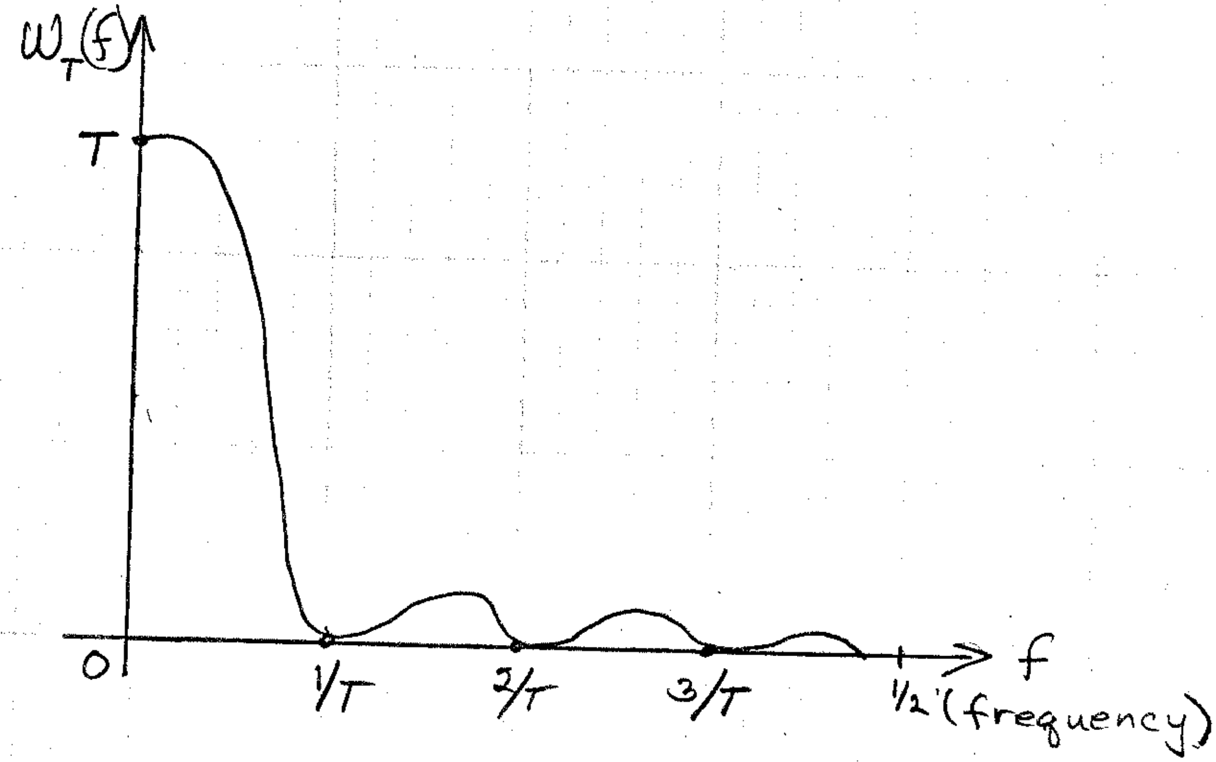
$$(2.5) \qquad \nu_{jl}(\tau) = \int_{-1/2}^{1/2} S_{jl}(f)e^{i2\pi f\tau}df$$

where $i = \sqrt{-1}$. That is we can replace the elements of $\Gamma(\tau)$ by the integrals in (2.5). Since S_{jl} does not depend on τ, a typical term in the summation (2.3) is

$$(2.6) \qquad q_j q_l \int_{-1/2}^{1/2} S_{jl}(f) \sum_{\tau=-T}^{T} (1-|\tau|/T)e^{i2\pi f\tau}df$$



2a: Bartlett's window in the time dimension



2b: Bartlett's window in the frequency domain

Fig. 2

It may be shown that

$$(2.7)\qquad \sum_{\tau=-T}^{T} (1-|\tau|/T)e^{i2\pi f\tau} = W_T(f) = \frac{\sin^2 T\pi f}{T\sin^2 \pi f}$$

for $-1/2 \leq f \leq 1/2$.*

Now $W_T(f)$ is Bartlett's window in the frequency domain, where, as Figure 2b suggests, it acts as a low-pass filter: low-frequency components enter the portfolio variance virtually unaltered, but high-frequency components are filtered out. It follows that most of the variation (2.3) comes from prices (or average returns) which exhibit cyclical behavior with low frequencies, i.e., long periods. Thus an optimal portfolio mix will tend to emphasize commodities with prices which exhibit short periodic variability as this variability will be attenuated by Bartlett's window.

2.3 Optimizing Rules

Let $\mu' = E(p_t)$ be a vector of (time-invariant) expected prices. Generalizing the procedure of section I, we wish to maximize

$$(2.8)\qquad \int_1 = T\cdot q'\cdot\mu - \alpha \mathrm{Var}R_T - \lambda_1(q'\cdot i-1)$$

subject to a non-negativity constraint $q \geqq 0$. T is the planning period, α is the cost associated with variance, λ_1 is a scalar Lagrange multiplier for the normalizing constraint, and i is a column vector of n units.

The first-order conditions are

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* The derivation is outlined in the appendix to this paper.

The authors are grateful to Professor E. P. Howrey for clarification of several points in this section.

(2.9) $$\frac{\partial \mathcal{L}_1}{\partial q} = T.\mu-\alpha[\sum_{\tau=-T}^{T}(T-|\tau|)(\Gamma(\tau)+\Gamma'(\tau)].q - \lambda_1 i \leqq 0$$

and

(2.10) $$\frac{\partial \mathcal{L}_1}{\partial \lambda_1} = -q' \cdot i + 1 \leqq 0$$

where equality holds if a particular q_i or λ_1 is positive.

Define the matrix

(2.11) $$B_T = \sum_{\tau=-T}^{T} (T-|\tau|)[\Gamma(\tau) + \Gamma'(\tau)]$$

so that (2.9) yields (assuming $q > 0$)

(2.12) $$q = B_T^{-1}(T \cdot \mu - \lambda_1 i)/\alpha \quad .$$

Substituting (2.12) into (2.10),

(2.13) $$\lambda_1 = \frac{T \cdot i' \cdot B_T^{-1} \cdot \mu - \alpha}{i' B_T^{-1} i} \quad .$$

Thus the optimal solution for q and λ_1 satisfies (2.10) and (2.11). These equations are easily solved.

For a second optimizing rule, we assume that there is a riskless asset which is held as insurance to cover a portion B of accumulated fluctuations in returns. Let $\rho = 1/(1+r)$ be the discount factor used where r is the rate of return on the riskless asset. The problem is then

$$(2.14) \qquad \text{Maximize } \mathcal{J}_2 = \sum_{t=1}^{T} \rho^{t-1} \cdot q' \cdot \mu - B \sum_{t=1}^{T} \rho^{t-1} \, \text{Var} R_t - \lambda_2 \, [q' \cdot i - 1]$$

where the first term is the discounted sum of expected revenues, the second term is the present value of the riskless asset required to cover a fraction B of the accumulated fluctuations, and λ_2 is a scalar Lagrange multiplier on the resource constraint.

Denoting

$$(2.15) \qquad B_m = \sum_{\tau=-m}^{m} (m - |\tau|)[\Gamma(\tau) + \Gamma'(\tau)]$$

for m = 1,...,T, we have the first order equations

$$(2.16) \qquad \frac{\partial \mathcal{J}_2}{\partial q} = \left(\frac{1-\rho T}{1-\rho} \right) \cdot \mu - B \cdot \left[\sum_{m=1}^{T} \rho^{m-1} B_m \right] \cdot q - \lambda_2 i \leqq 0$$

and

$$(2.17) \qquad \frac{\partial \mathcal{J}_2}{\partial \lambda_2} = -q \cdot i + 1 \leqq 0$$

where equality holds if q_i or λ_2 is positive.

The solution equations are then

$$(2.18) \qquad q = \frac{1}{B}\left[\sum_{m=1}^{T} \rho^{m-1} B_m\right]^{-1} \cdot \left[\left(\frac{1-\rho^{T}}{1-\rho}\right)\cdot\mu - \lambda_2 \cdot i\right]$$

and

$$(2.19) \qquad \lambda_2 = \frac{i' \cdot \left[\sum_{m=1}^{T} \rho^{m-1} B_m\right]^{-1} \cdot i \cdot \left(\frac{1-\rho^{T}}{1-\rho}\right)\cdot\mu - B}{i'\left[\sum_{m=1}^{T} \rho^{m-1} B_m\right]^{-1} i}$$

if $q > 0$.

This solution is, of course, very similar to the first. In this case the matrix B_T is replaced by a discounted sum of B_m matrices whose elements are auto- and cross-covariance functions filtered by Bartlett's window. As the index m of the matrix B_m increases, the window increasingly accentuates low-frequency components of variance and attenuates high-frequency components. Thus one expects that as the discount rate r increases, the low-frequency components will become less potent in terms of contribution to variance and the optimal portfolio will move relatively more towards assets with long-cycle (low-frequency) behavior.

Second-order conditions are easily obtained by noting that the B_m matrices are positive definite.

III. Empirical Tests of the Model

As mentioned above, we would expect that commodities whose prices exhibit long swings in prices even though they may have a relatively small total variance about the mean are likely to be less favored in an optimal portfolio. Furthermore, long swing commodities are likely to become less desirable as the planning period increases, as the discount rate declines, or as the cost associated with risk increases. As a rough empirical check of the importance of these factors, we computed auto- and cross-covariances for world prices of 9 primary commodities (U.S. dollars per ton) on a quarterly basis for the period 1955 to 1968 [2]. The computations were based on the assumption that each price followed an Nth order autoregressive process

$$(3.1) \qquad (p_{it}-\mu_i) = \sum_{S=1}^{N} \alpha_{is}(p_{it-S}-\mu_i) + Z_{it}$$

where p_{it} is the price of the ith commodity in period t, μ_i is the expected price of the ith commodity, and Z_{it} is a disturbance term with expectation zero, no autocorrelation, and variance constant over time. The disturbance terms for different commodities, Z_{it} and Z_{jt}, were assumed contemporaneously correlated but independently distributed about the non-zero lags.

The order N of the auto-regressive process was determined by least-squares estimation of autoregressive structures of successively higher orders until the disturbance term Z_{it} was not significantly autocorrelated. The estimated equations for the 9 commodities are shown in Table 1.

Table 1

Autoregressive Price Equations

			R^2	D.W.
1.	Burlap	$p_{1t} = 0.7\, p_{1,t-1} + Z_{1t}$	0.82	1.86
2.	Cocoa	$p_{2t} = 1.34\, p_{2,t-1} - 0.42\, p_{2,t-2} + Z_{2t}$	0.83	1.70
3.	Coconut Oil	$p_{3t} = 0.82\, p_{3,t-1} + Z_{3t}$	0.67	1.55
4.	Copra	$p_{4t} = 0.83\, p_{4,t-1} + Z_{4t}$	0.70	1.60
5.	Cotton	$p_{5t} = 0.76\, p_{5,t-1} + Z_{5t}$	0.80	2.25
6.	Hemp	$p_{6t} = 0.88\, p_{6,t-1} + Z_{6t}$	0.87	1.60
7.	Tea	$p_{7t} = 0.9\, p_{7,t-1} - 0.3\, p_{7,t-2} + Z_{7t}$	0.63	2.00
8.	Tin	$p_{8t} = 0.85\, p_{it-1} + Z_{8t}$	0.92	1.97
9.	Wool	$p_{9t} = 1.32\, p_{9,t-1} - 0.46\, p_{9,t-2} + Z_{9t}$	0.85	2.00

Note: Prices are measured per ton as deviations from the following means:

Burlap: $\bar{p}_1$ = \$11.13, Copra: $\bar{p}_4$ = \$9.12, Tea: $\bar{p}_7$ = \$50.3,

Cocoa: $\bar{p}_2$ = \$28.05, Cotton: $\bar{p}_5$ = \$26.35, Tin: $\bar{p}_8$ = \$125.22,

Coconut Oil: $\bar{p}_3$ = \$15.60, Hemp: $\bar{p}_6$ = \$24.37, Wool: $\bar{p}_9$ = \$56.64

The matrices $\Gamma(\tau)$ of auto- and cross-covariances were computed in two steps. First the autocovariances were estimated from recursion equations:

$$(3.2) \qquad \hat{\nu}_{11}(1) = \frac{\hat{\alpha}_{11}\hat{\nu}_{11}(0)}{1-\hat{\alpha}_{12}}$$

and

$$(3.3) \qquad \hat{\nu}_{11}(\tau) = \hat{\alpha}_{11}\hat{\nu}_{11}(\tau-1) + \hat{\alpha}_{12}\hat{\nu}_{11}(\tau-2)$$

for $\tau = 2, \ldots$. In these equations $\hat{\nu}_{11}(0)$ is the usual estimate of the variance and $\hat{\alpha}_{11}$, $\hat{\alpha}_{12}$ are the least-squares estimates of Table 1. (For a first-order process, $\hat{\alpha}_{12} = 0$.)

Next, the same least-squares autoregressive parameters were used to compute maximum-likelihood estimates $\hat{\nu}_{ij}(\tau)$ of the cross-covariance between commodity i and commodity j at lag τ for m observations:

$$(3.4) \qquad \hat{\nu}_{ij}(\tau) = \frac{1}{m} \sum_{t=1}^{M-\tau} (p_{it} - \sum_{S=1}^{N} \hat{\alpha}_{is} p_{it-S}) \cdot (p_{jt+\tau} - \sum_{S=1}^{N} \hat{\alpha}_{jS} p_{t-\tau-S}) \quad .$$

The maximum-likelihood procedures used here are discussed in Jenkins and Watts [3, pp. 189-192, 338-340]. The logic for formula (3.4) is that the autoregressive processes must be reduced to white noise before the cross-covariance functions can be estimated. Otherwise large auto-covariances would give rise to spurious cross-covariances.

The autoregressive parameters $\hat{\alpha}_{iS}$ and the estimated variances of the disturbance terms $\hat{\sigma}^2_{Zi}$ may be used to compute the spectrum component of variance $S_{ii}(f)$ of commodity i at any frequency f according to the formula

$$(3.5)\quad \hat{S}_{ii}(f) = \frac{2\hat{\sigma}^2_{Zi}}{1 + \hat{\alpha}^2_{i1} + \hat{\alpha}^2_{i2} - 2\hat{\alpha}_{i1}(1-\alpha_{i2})\cos 2\pi f - 2\hat{\alpha}_{i2}\cos 4\pi f}$$

for $0 \leq f \leq 1/2$ (cycles per quarter).
By integrating (3.5) over frequency for each commodity we can determine the proportion of total variance at or below any frequency. Table 2 gives the cumulative distribution of variance for each of the 9 commodities; and it is seen that burlap, cotton, and tea prices are relatively short swing while cocoa and hemp have long-swing prices. We would therefore expect that the commodity mix would shift toward burlap, cotton, or tea as we go from static portfolio analysis to dynamic analysis.

We now tabulate optimal output levels for export portfolios composed of some of the commodities in Table 1. Static and dynamic optimal mixes are discussed for a range of values of the parameters α (the cost of variance and ρ (the discount rate).

Table 3 gives some solutions to the portfolio problem which is to maximize (2.7) (no discount rate used) with the following five commodities:

1. Cocoa (long swing)
2. Hemp (long swing)
3. Tea (short swing)
4. Tin (long swing, large variance)
5. Wool (intermediate swing) .

In order to solve this problem, we used the non-linear programming algorithm due to Houthakker [4]. Two sets of solutions are given

Table 2

Cumulative Distribution of Variance

Frequency (cycles/quarter)	(1) Burlap	(2) Cocoa	(3) Coconut Oil
0.0	.28	.70	.50
0.05	.60	.98	.80
0.10	.75	.99	.99
0.15	.82	.99+	.99+
0.5	1.0	1.0	1.0

Frequency	(4) Copra	(5) Cotton	(6) Hemp
0.0	.54	.37	.78
0.05	.82	.69	.99
0.10	.91	.81	.99+
0.15	.95	.87	.99+
0.5	1.0	1.0	1.0

Frequency	(7) Tea	(8) Tin	(9) Wool
0.0	.15	.62	.37
0.05	.44	.88	.81
0.10	.70	.96	.93
0.15	.83	.99	.99
0.5	1.0	1.0	1.0

Entries give per cent of total variance at or below the indicated frequency.

Table 3

Optimal Portfolios for Cocoa, Hemp, Tea, Tin and Wool*

Classical Static Case						Filtered Dynamic Case				
Commodities						Commodities				
1	2	3	4	5	α	1	2	3	4	5
0	0	0	0.81	0.19	0.05	0	0	0.53	0.12	0.35
0	0	0.016	0.43	0.55	0.10	0	0	0.65	0.06	0.29
0	0	0.21	0.29	0.50	0.15	0	0	0.68	0.05	0.27
0	0	0.30	0.23	0.47	0.20	0.025	0.02	0.67	0.04	0.25
0	0	0.36	0.18	0.46	0.25	0.037	0.08	0.63	0.03	0.23

* Planning horizon T = 16 quarters

in Table 3. The cost of variance α is varied from 0.05 to 0.25. Note that the short swing commodity 3 and intermediate swing commodity 5 increase in importance as we move from the static to dynamic case with $\alpha = 0.05$. For larger α, the short swing commodity 3 increases in the optimal portfolio. The long swing commodity 4 declines drastically as we move from the static to dynamic case for all values of α. As α increases, the large variance commodity 4 decreases in importance in both the static and dynamic cases.

Table 4 shows the same commodities in optimal portfolios for the static and dynamic cases. The optimizing problem is given by (2.14) and assumes that a riskless asset is held to cover a certain proportion of accumulated fluctuations in returns. In the classical case, the changes in the optimal portfolio are slight. In the dynamic case, the changes in optimal portfolios are more pronounced. The short swing commodity 3 declines in importance while the long swing commodity 4 increases in importance. The reason for this phenomenon is that the cost of long swings in variance is attenuated by the higher discount rate.

Table 4

Optimal Portfolios for Different Discount Rates*

Classical Static Case: Commodities 1	2	3	4	5	ρ	Filtered Dynamic Case: Commodities 1	2	3	4	5
0	0	0	0.45	0.55	1.05	0	0	0.60	0.08	0.32
0	0	0	0.46	0.54	1.17	0	0	0.51	0.13	0.36
0	0	0	0.52	0.48	1.32	0	0	0.35	0.21	0.44

* $\alpha = 0.05$, planning horizon = 16 quarters.

References

[1] W. C. Brainard and R. N. Cooper, "Uncertainty and Diversification in International Trade." Cowles Foundation Discussion Paper No. 197, New Haven, October 27, 1965. Subsequently published as Food Research Institute Study 8, No. 3, 1968, 257-85.

[2] International Monetary Fund, International Financial Statistics, various issues. The data are spot commodity prices or, occasionally, unit values.

[3] G. M. Jenkins and D. G. Watts, Spectral Analysis and Its Applications. San Francisco: Holden-Day, 1968.

[4] H. S. Houthakker, "The Capacity Method of Quadratic Programming," Econometrica, 28, 1960, pp. 62-87.

APPENDIX

To outline the derivation of (2.7) we first note the following trigonometric identities (where $\omega = 2\pi f$):

(A.1) $$\sum_{t=1}^{T} \cos\omega t = \frac{\cos\left(\frac{T+1}{2}\right)\omega \sin\frac{T\omega}{2}}{\sin\frac{\omega}{2}}$$

(A.2) $$2\cos\frac{1}{2}(T\omega + \omega)\sin\frac{1}{2}(T\omega) = \sin\left(T\omega + \frac{\omega}{2}\right) - \sin\left(\frac{\omega}{2}\right)$$

(A.3) $$\sum_{t=1}^{T} \sin\omega t = \frac{\sin\frac{T+1}{2}\omega \sin\frac{T\omega}{2}}{\sin\frac{\omega}{2}}$$

(A.4) $$2\sin\frac{1}{2}(T\omega + \omega)\sin\frac{1}{2}(T\omega) = -\left[\cos\left(T\omega + \frac{\omega}{2}\right) - \cos\frac{\omega}{2}\right]$$

(A.5) $$\sin\frac{T\omega}{2} = \pm\sqrt{\frac{1}{2} - \frac{1}{2}\cos T\omega}\ .$$

Use (A.1) and (A.2) to show that

(A.6) $$\sum_{t=-T}^{T} e^{i\omega t} = \frac{\sin\left(T + \frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}\ .$$

Use (A.3) and (A.4) to show that

$$\text{(A.7)} \qquad 2 \sum_{t=1}^{T} \sin \omega t = \frac{-[\cos(T + \frac{1}{2})\omega - \cos \frac{\omega}{2}]}{\sin \frac{\omega}{2}} \quad .$$

Use (A.6) and (A.7) to show that

$$\sum_{t=-T}^{T} (1 - \frac{|t|}{T}) e^{i\omega t} = \sum_{t=-T}^{T} e^{i\omega t} - 2T^{-1} \sum_{t=1}^{T} t\cos\omega t$$

$$\text{(A.8)} \qquad = \frac{\frac{1}{2} - \frac{1}{2}\cos T\omega}{T\sin^2 \frac{\omega}{2}} \quad .$$

In this step use the fact that

$$2 \sum_{t=1}^{T} t\cos\omega t = \frac{2d}{d\omega} \sum_{t=1}^{T} \sin\omega t = \frac{-\frac{1}{2} + T\sin\frac{\omega}{2} \sin (T + \frac{1}{2})\omega + \frac{1}{2}\cos T\omega}{\sin^2 \frac{\omega}{2}} \quad .$$

Use (A.5) in (A.8) to complete the proof that

$$\sum_{t=-T}^{T} (1 - \frac{|t|}{T}) e^{i\omega t} = \frac{\sin^2 \frac{T\omega}{2}}{T\sin^2 \frac{\omega}{2}} \quad .$$